Decision Trees

The systematic use of trees for knowledge representation can be used for fast and frugal decisions.

Tree-structured schemes are ubiquitous tools for organising and representing knowledge.

Bayesian model

Probabilistic Modelling

-A model describes data that one could observe from a system,

-If we use the mathematics of probability theory to express all forms of uncertainty and noise associated with our model...

- ...then inverse probability (i.e. Bayes rule) allows us to infer unknown quantities, adapt our models, make predictions and learn from data.

Bayes Rule

$$P(\text{hypothesis}/\text{data}) = \frac{P(\text{data}/\text{hypothesis})P(\text{hypothesis})}{P(\text{data})}$$

Rev'd Thomas Bayes (1702-1761)

Bayes rule tells us how to do inference about hypotheses from data.

Learning and prediction can be seen as forms of inference.

Bayesian inference grows out of a simple formula known as Bayes' rule (Bayes, 1763/1958). When stated in terms of abstract random variables, Bayes' rule is no more than an elementary result of probability theory. Assume we have two random variables, A and B.

One of the principles of probability theory (sometimes called the chain rule) allows us to write the joint probability of these two variables taking on particular values *a* and b, P(a, b), as the product of the conditional probability that A will take on value a given B takes on value b, P(a|b), and the marginal probability that B takes on value b, P(b).

Thus, we have
$P(a, b) = P(a|b)P(b)$. (1)
There was nothing special about the choice of A rather than B in factorizing the joint probability in this way, so we can also write $P(a, b) = P(b|a)P(a)$. (2)
It follows from Equations 1 and 2 that $P(a|b)P(b) = P(b|a)P(a)$, which can be rearranged to give $P(b|a)$
$= \quad P(a|b)P(b)$

$P(a)$. (3)

This expression is Bayes' rule, which indicates how we can compute the conditional probability of b given a from the conditional probability of a given b.

System analysis and decision making

The both full Bayesian inference and one-reason decision making are processes that can be described in terms of tree-structured decision rules.

A fully specified Bayesian model can be represented by means of the “full” or “maximal” tree obtained by introducing nodes for all conceivable conjunctions of events, whereas one-reason decision rule can be represented by a “minimal” subtree of the maximal tree (with maximal and minimal reference to the number of paths connecting the root to the leaves).

Subtrees of the full tree not containing any path from a root to leaves are regarded as “truncated” since they necessarily truncate the access to available information.

Minimal trees can be obtained by radically pruning the full tree. A minimal tree has a leaf at each one of its levels, so that every level allows for a possible decision.

Indeed, when a radical reduction of complexity is necessary and when the environment is favorable, such a minimal tree will be extremely fast and frugal with negligible losses in accuracy.

A name for such minimal trees is "fast and frugal trees".

TREE-STRUCTURED REPRESENTATIONS IN CLASSIFICATION TASKS

System analysis and decision making

Human classifications and decisions are based on the analysis of features or cues that the mind/brain extracts from the environment.

There is a wide spectrum of classification schemes, varying in terms of the time scale they require, from almost automatic classifications the mind/brain performs without taking real notice, up to slow, conscious ones.

Among the diverse representation a device for classification, trees have been the most ubiquitous.

Since the fourth century, trees representing sequential step-by-step processes for classification based on cue information have been common devices in many realms of human knowledge.

These trees start from a root node and descend through branches connecting the root to intermediate nodes until they reach final nodes or leaves.

A classification (also called categorization) tree is a graphical representation of a rule - or a set of rules - for making classifications.

Each **node** of the tree represents a question regarding certain features of the objects to be classified or categorized.

Each **branch** leading out of the node represents a different answer to the question.

It is assumed that **the answers to the question are exclusive** (non-overlapping) and **exhaustive** (cover all objects).

That is, there is exactly one answer to the question for each object, and each of the possible answers is represented by one branch out of the node.

The nodes below a given node are called its “children”, and the node above a node is called its “parent”.

Every node has exactly one parent except for the “root” node, which has no parent, and which is usually depicted at the top or far left. The “leaf” nodes, or nodes having no children, are usually depicted at the bottom or far right.

In a **"binary" tree**, all non-leaf nodes have exactly two children; in general trees nodes may have any number of children.

The leaf nodes of a classification tree represent a "**partition" of the set of objects** into classes defined by the answers to the questions.

Each leaf node has an associated class label, to be assigned to all objects for which the appropriate answers are given to the questions associated with the leaf's ancestor nodes.

System analysis and decision making

The classification tree can be used to construct a simple algorithm for associating any object with a class label.

Given an object, the algorithm traversesa “path” from the root node to one of the leaf nodes. This path is determined by the answers to the questions associated with the nodes.

The questions and answers can be used to define a “decision rule” to be executed when each node is traversed. The decision rule instructs the algorithm which arc to traverse out of the node, and thus which child to visit.

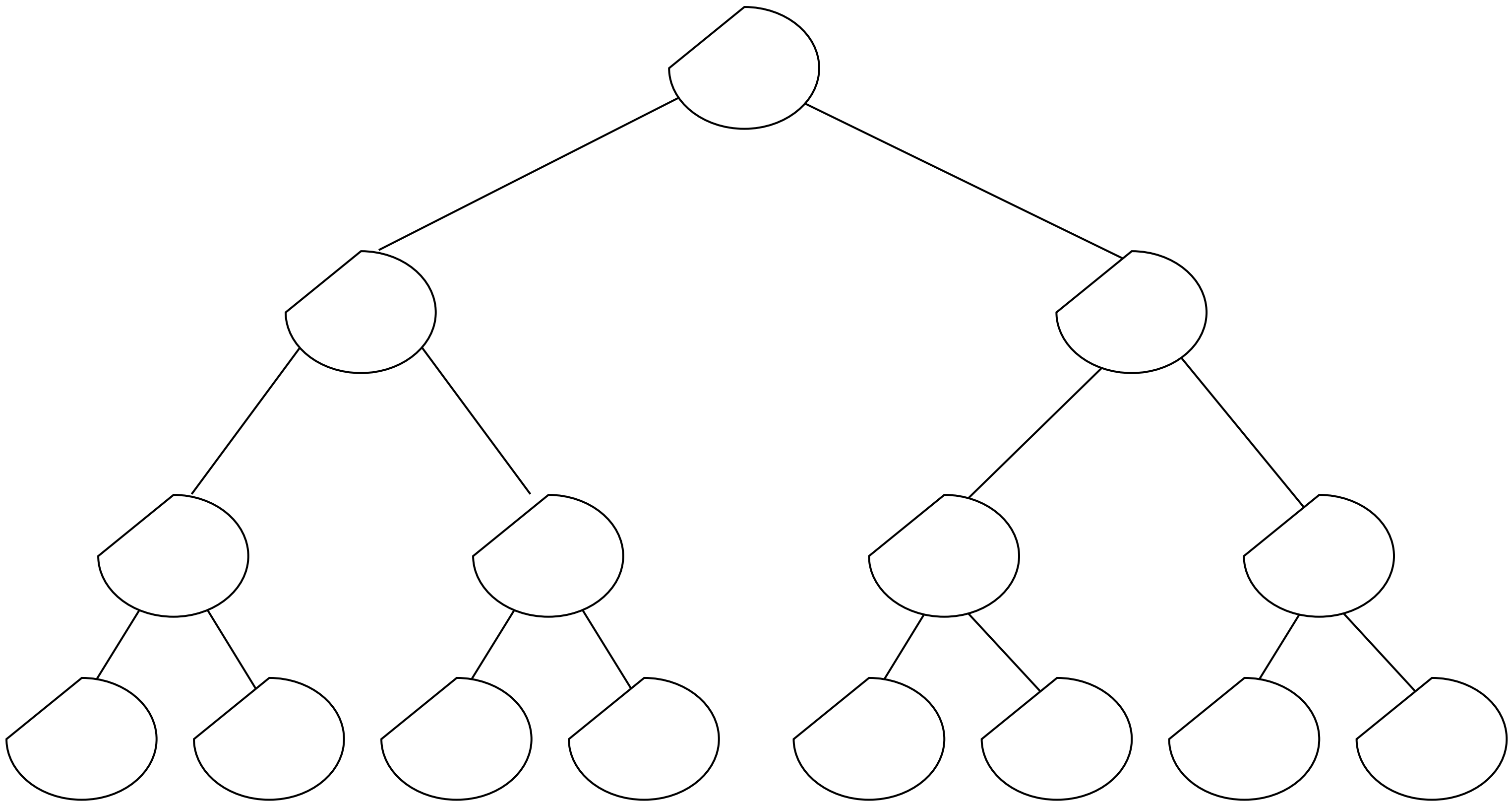
Algorithm TREE-CLASS:

1. Begin at root node.
2. Execute rule associated with current node to decide which arc to traverse.
3. Proceed to child at end of chosen arc.
4. If child is a leaf node, assign to object the class label associated with node and STOP.
5. Otherwise, go to (2).

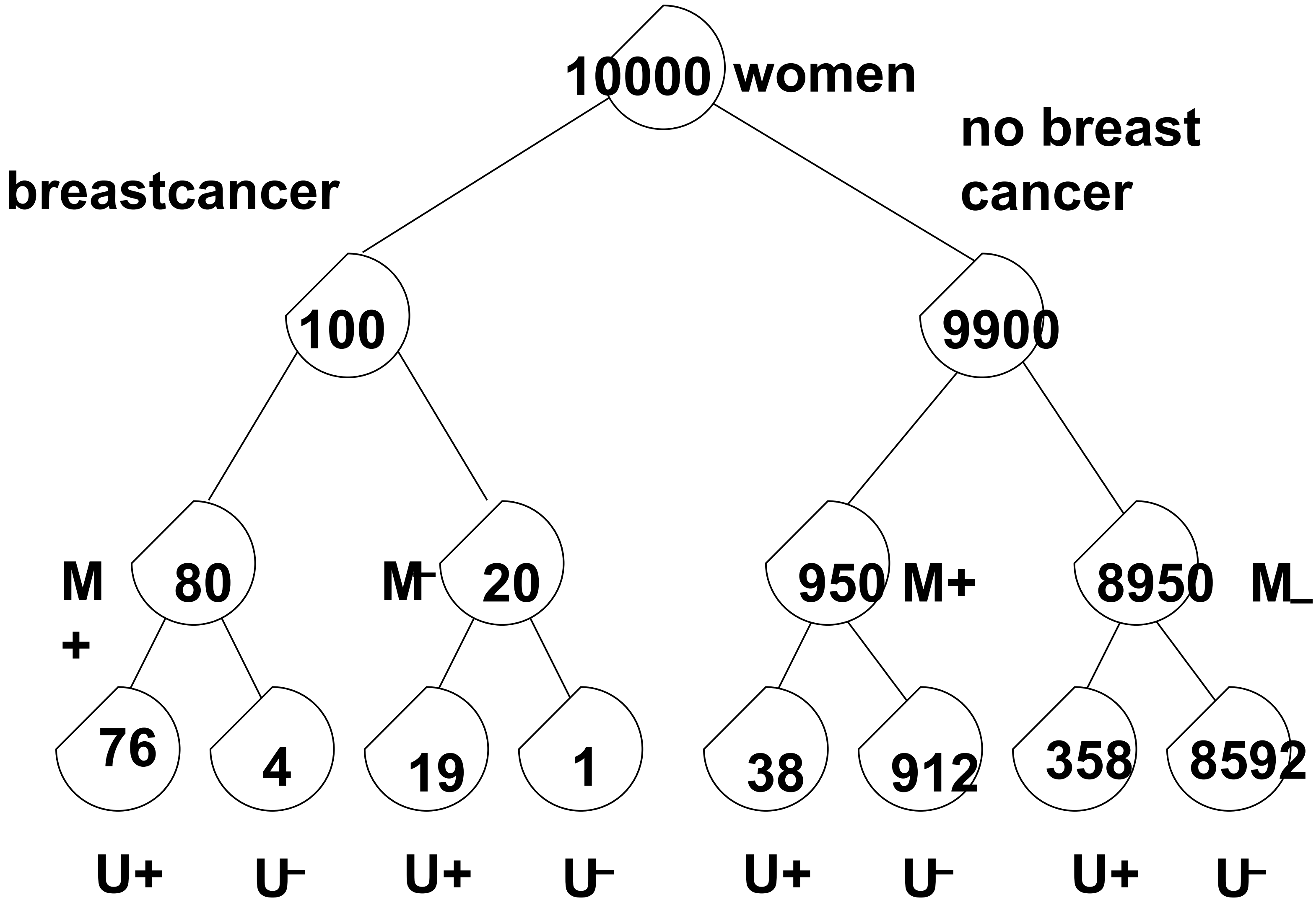
Natural Frequency Trees

Natural frequency trees provide good representations of the statistical data relevant to the construction of optimal classification trees.

System analysis and decision making



Figures 1. The natural frequency tree for classifying a patient as having or not having cancer, based on the results of a mammogram and an ultrasound test



How many of the women who get a positive mammography and a positive ultrasound test do you expect to actually have breast cancer?

"Natural frequency tree".

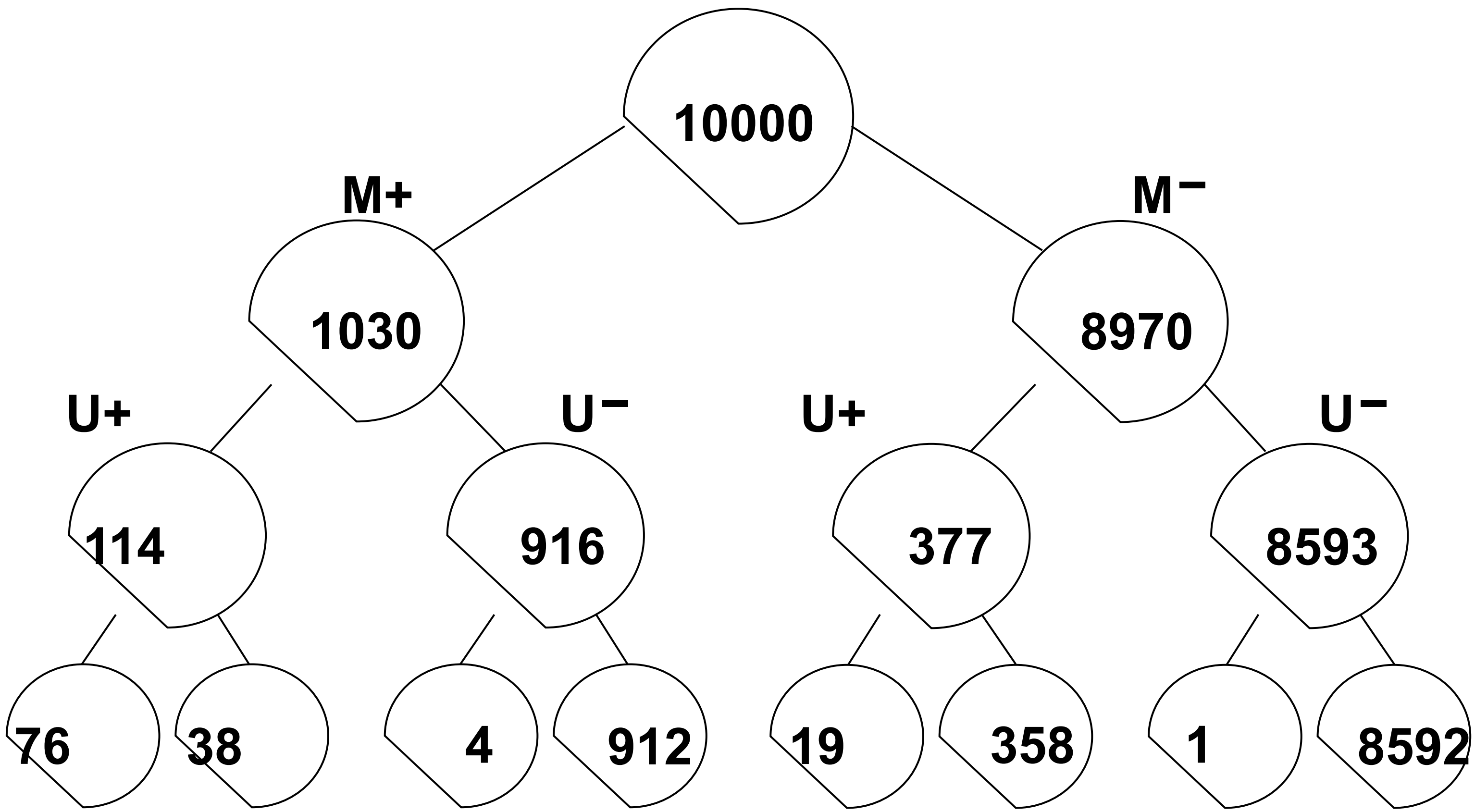
The numbers in the nodes indicate that the two tests are conditionally independent, given cancer.

This is obviously an assumption the reality of medical tests is that neither combined sensitivities nor combined specificities are reported in the literature.
It is a frequent convention to assume tests' conditional independence, given the disease.

There are more practical natural frequency trees for diagnosis.

They are obtained by inverting the order followed for the sequential partitioning of the total population (10000 women)

Figures 2. The natural frequency tree obtained from the tree, when the sampling order is mammogram → ultrasound → cancer



Organizing the tree in the *diagnostic* direction produces a much more efficient classification strategy.

This tree has two major advantages over the tree in Figure 1. for a diagnostic task.

First, we can follow the TREE-CLASS algorithm for the first two steps before becoming stuck at the second-to-last level above the leaf nodes.

For example, for the hypothetical woman with M+ and U+ described above, we would be able to place her among the 114 women at the leftmost node on the third level from the top.

Second, once we have placed a patient at a node just above the bottom of the tree, we can compute the probability of placing her at each of the two possible leaf nodes by using only local information.

That is, the probability comparing the leaves of Figures 1 and 2 reveals that they are the same.
That is, they contain the same numbers, although their ordering is different, as is the topology of their connection to the rest of the tree.
One might question whether a natural sampler would partition the population in the causal or the diagnostic direction.

Knowledge tends to be organised causally, and diagnostic inference is performed by means of inversion strategies, which, in the frequency format, are reduced to inventing the partitioning order as above

(in the probability format, the inversion is carried out by applying Bayes' theorem).

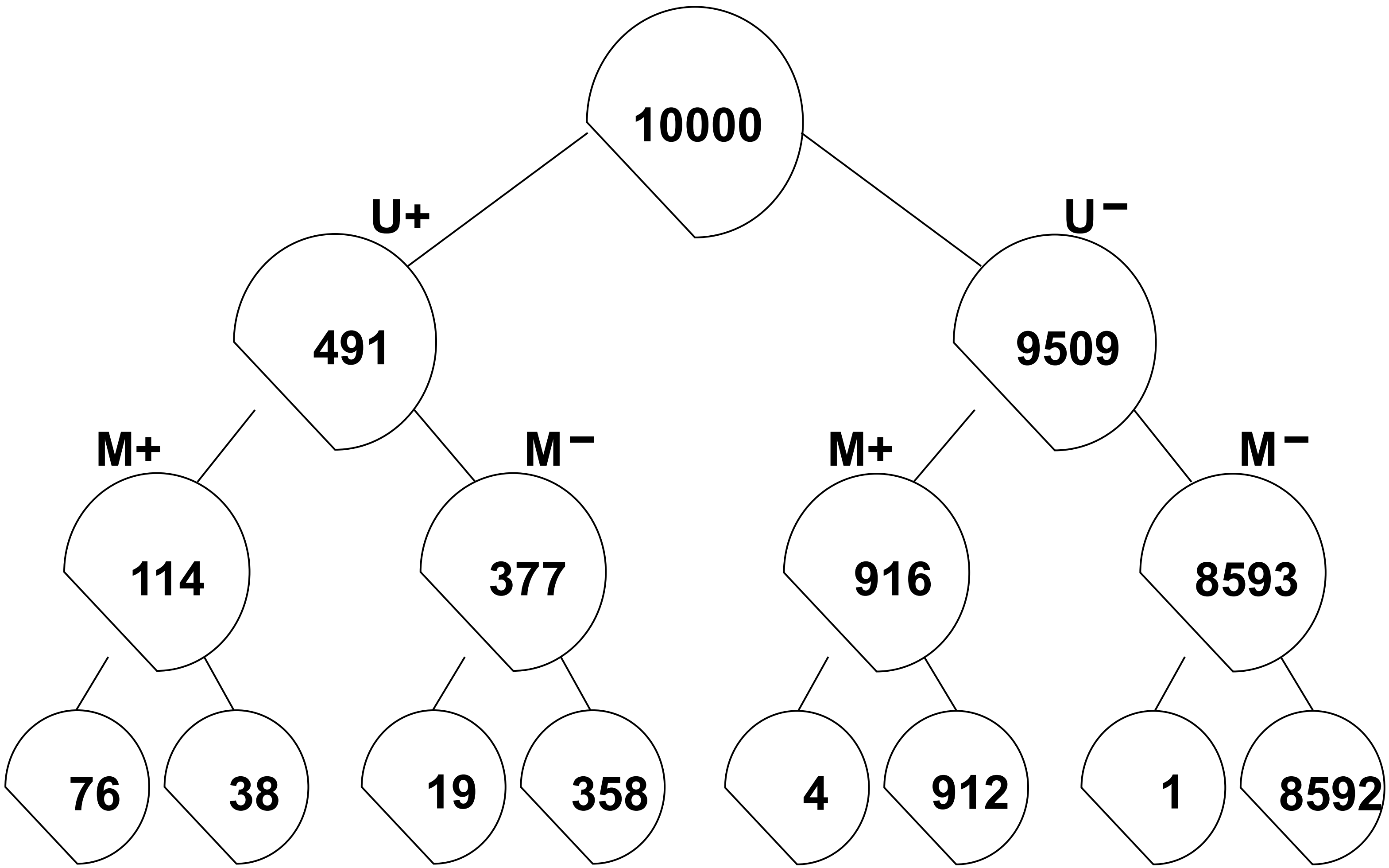
However, ecologically situated agents tend to adopt representations tailored to their goals and the environment in which they are situated.

Thus, it might be argued that a goal-oriented natural sampler performing a diagnostic task will probably partition the original population according to the cues first, and end by partitioning according to the criterion.

Now, consider another version of the diagnostic ordering of the cues, where, **in the first phase**, women are partitioned according to their ultrasound, and **in the second phase,** they are partitioned according to the mammograms and finally according to breast cancer.

The tree is depicted in Figure 3.

Figure 3. Natural sampling in the order ultrasound → mammography → cancer our hypothetical woman has cancer can be computed by looking at the cancer node just below, discovering that there are 76 exemplars associated with that node, and dividing it by the 114 exemplars at the third level.



FAST AND FRUGAL TREES

A tree may be called a fast and frugal tree, that is, trees constructed with binary cues and a binary criterion.

The generalisation to other cases is straightforward. With the classification according to a binary criterion (for example, “cancer” or “no cancer”), we associate two possible decisions, one for each possible classification (for example, “biopsy” or “no biopsy”).

System analysis and decision making

An important convention has to be applied beforehand: cue profiles can be expressed as vectors of 0s and 1s, where a 1 corresponds to the value of the cue more highly correlated with the outcome of the criterion considered “positive” (for example, a presence of cancer).

The convention is that left branches are labelled with 1s and right branches with 0s.

Thus, each branch of the fully specified tree can be labelled with a 1 or a 0, according to the cue value associated with the node at the end of the branch.

Definition

A fast and frugal binary decision tree is a decision tree with at least one exit leaf at every level. That is, for every checked cue, at least one of its outcomes can lead to a decision.

In accordance with the convention applied above, if a leaf stems from a branch labelled 1, the decision will be positive (for example, “perform biopsy”).

We begin by recalling that according to our convention, we will encode “having the disease” with a 1, and “not having the disease” with a 0.

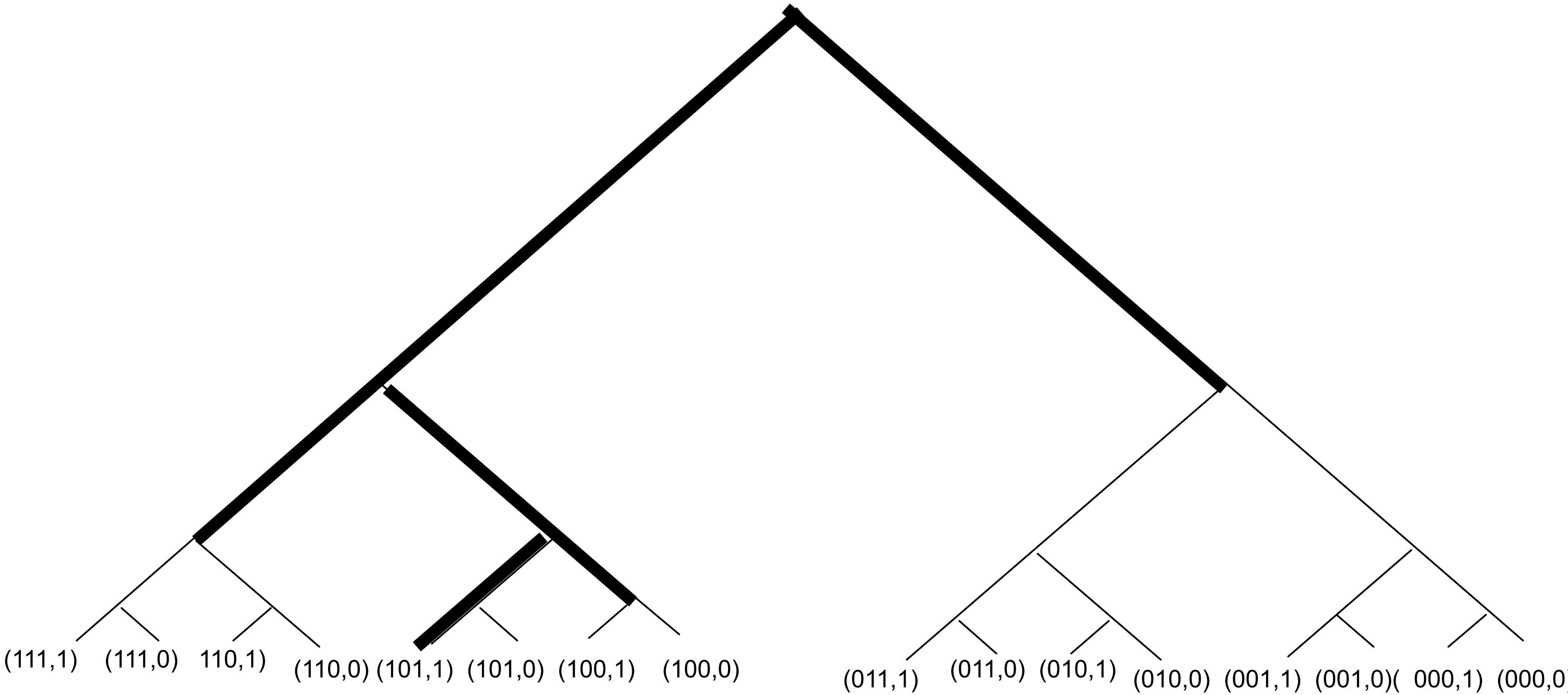
If we have, say, three cues, the leaves of the full frequency tree will be labeled (111,1), (111,0), (101,1), (101,0), (100,1), (100,0), (011,1), (011,0), (010,1), (010,1), (001,0), (000,1), (000,0), where the binary vectors will appear in decreasing lexicographic order from left to right.

Observe that the cue profile from the state of the disease is separated by a comma.

Since this ordering is similar to the ordering of words in a dictionary, it is usually called “lexicographic”.

Lexicographic orderings allow for simple classifications, by establishing that all profiles larger (in the lexicographic ordering) than a certain fixed profile will be assigned to one class, and all profiles smaller than the same fixed profile will be assigned to the other class.

A lexicographic classifier determined by the path of profile (101), where the three bits are cue values and the last bit corresponds to the criterion (for example, having or not having the disease)



A “lexicographic decision rule” makes one decision, say, D, for all profiles larger than a given, fixed profile, and the alternative decision, ¬D, for all profiles smaller than that same profile.

The profile itself is assigned decision D if it ends with a 1, and decision ¬D if it ends with a 0.

A fast and frugal decision tree makes decisions lexicographically. This is what we prove in the following theorem.

Constructing Fast and Frugal Decision Trees

Situation: A man is rushed to a hospital with severe chest pain. The doctors have to decide whether the patient should be assigned to the coronary care unit (CCU) or to a monitored nursing bed (NB).

The cues on which a doctor bases such a decision are the following:

(1)ST segment elevation in the electrocardiogram (ECG)

(2)patient report of chest pain as the most important symptom

(3)history of heart attack

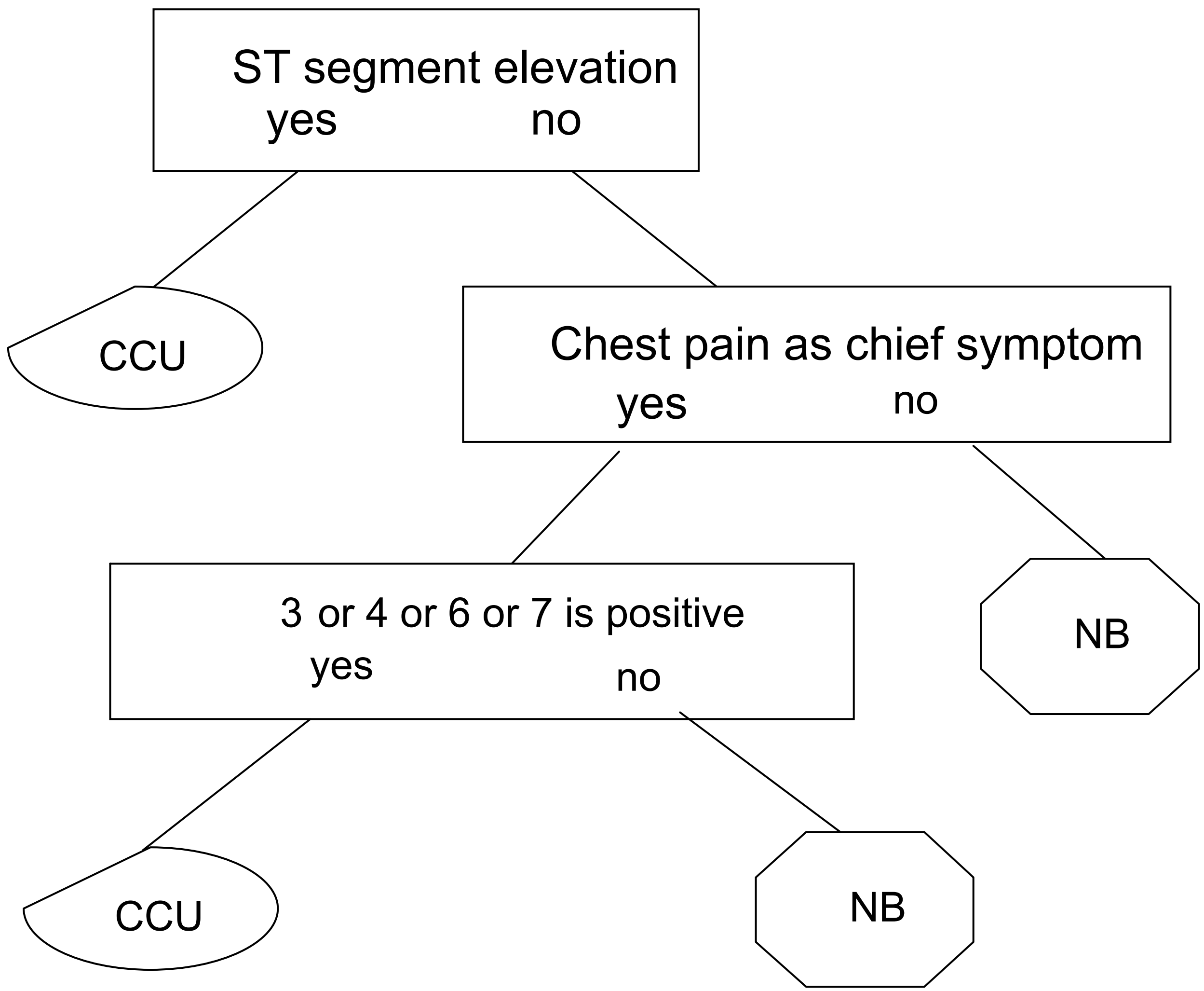
(4)history of nitroglycerin use for chest pain

(5)pain in the chest or left arm

(6)ST segment barring

(7)T-waves with peaking or inversion.

Green and Mehr (1997) analyzed the problem of finding a simple procedure for determining an action based on this cue information. They reduced the seven cues to only three (creating a new cue formed by the disjunction of 3, 4, 6 and 7) and proposed the tree depicted in Figure.



Although Green and Mehr (1997) succeeded in constructing a fast and frugal decision tree with excellent performance, they did not reveal how they ended up with precisely this tree, nor did they provide any standard procedure to construct such trees.

Our intention is to provide simple rules for their construction. Using the Green and Mehr task as an example, we will illustrate several methods for designing fast and frugal trees and then compare their performance.

In order to construct a fast and frugal tree, one can, of course, test all possible orderings of cues and shapes of trees on the provided data set and optimize fitting performance; in the general case, this requires enormous computation if the number of cues is large.

Another approach is to determine the “best” cue according to some given rule, and then determine the “second best” cue conditional on the first, and so on. But this again requires a fairly large number of computations.

In conceptual analogy to Bayes models, decision makers will not look into conditional dependencies and/or correlations between cues.

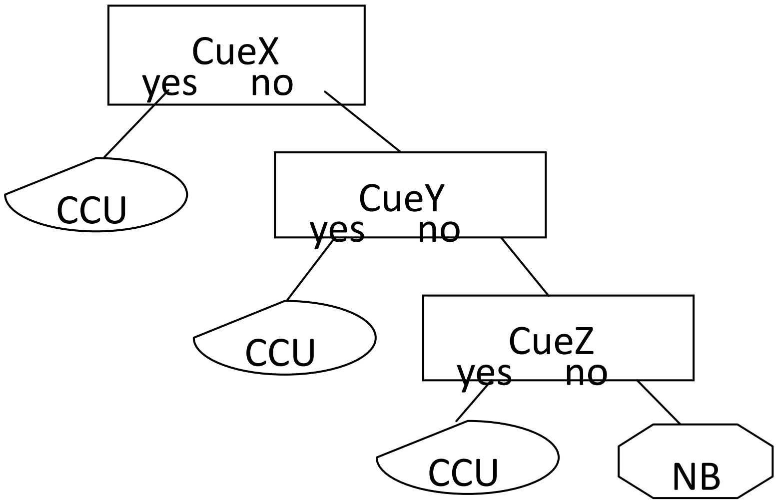
The question is: What is a good cue?

The Shape of Trees

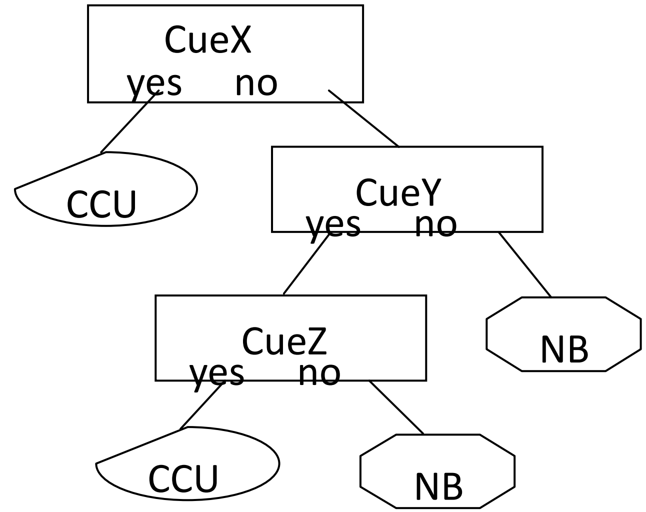
There are **four possible shapes**, or branching structures, of fast and frugal trees for three cues.

System analysis and decision making

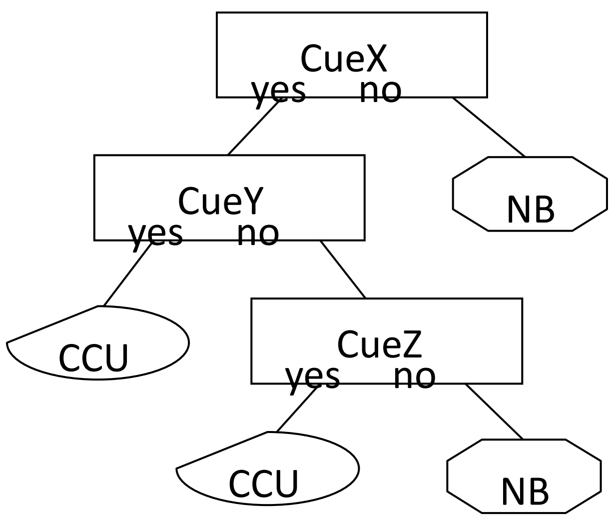
Type1



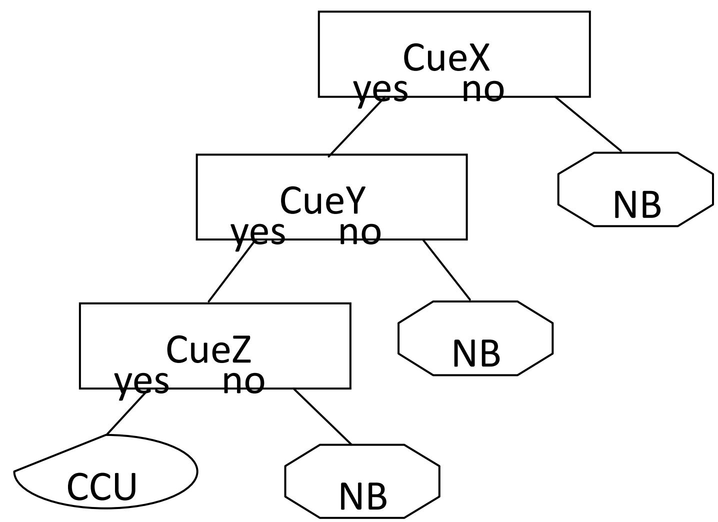
Type2



Type3



Type4



Trees of type 1 and 4 are called “rakes” or “pectinates”.

As defined here, rakes have a very special property.

They embody a strict conjunction rule, meaning that one of the two alternative decisions is made only if all cues are present (type 1) or absent (type 4).

Trees of types 2 and 3 are called "zigzag trees".

They have the property of alternating between positive and negative exits in the sequence of levels.

Cue interactions go beyond the bivariate contingencies that are typically observed in the naive (unconditional) linear model framework.

A straightforward demonstration of the interaction effect is given by what is now called “Meehl’s paradox” (after its initial description by the clinician-statistician Paul E. Meehl, one of the pioneers in the field of clinical decision making, 1950).

Meehl's paradox in the binary case

Criterion	Cue 1	Cue 2
1	1	1
1	0	0
1	1	1
1	0	0
1	1	1
1	0	0
0	0	1
0	1	0
0	0	1
0	1	0
0	0	1
0	1	0

The “paradoxical” nature of the given example is due to the fact that both single cues are essentially uncorrelated with the criterion from a bivariate perspective.

Note also that the intercorrelation between cues is 0. Still, both cues together allow a perfect prediction of the criterion: the criterion value is present when both cues are either present or absent (the {11} and {00} cases), and absent if only one of them is present (the {10} and {10} cases, respectively).

Both cues observed simultaneously contain predictive information that cannot be decomposed into an “additive” bivariate view. The dual-cue pattern cannot be reduced to the contributions of either cue alone.

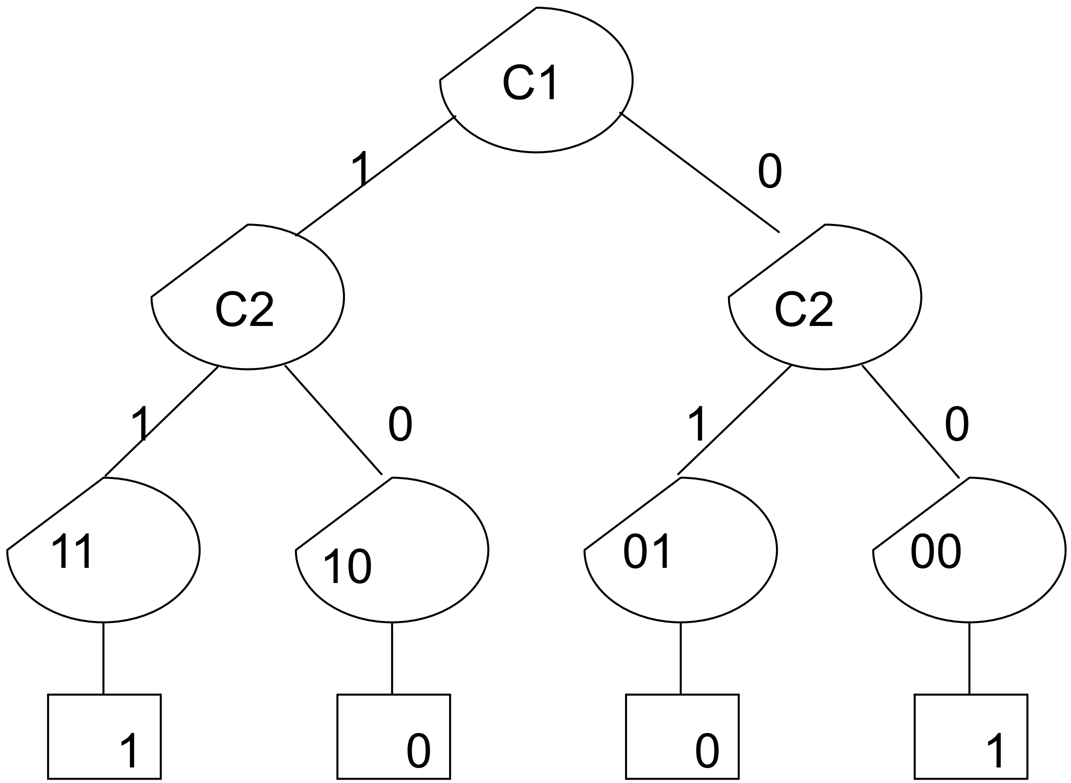
Correlations between cue 1 and the criterion in manifest subclasses indicated by cue 2

For cue 2 = 0	For cue 2 = 1
0 3	3 0
3 0	0 3

Another way to put it is to look at one of the two cues as a classifier that discriminates between those cases where the correlation between the other cue and the criterion is positive and those where it is negative.

The dataset is a mixture of cases with either positive or negative intercorrelation between one cue and the criterion, with the other cue indicating the type of contingency

Representation of Meehl's paradox in a full tree



Simple trees bet on a certain structure of the world, irrespective of the small fluctuations in a given set of available data.

This can be a major advantage for generalisation if the stable part of the process, which also holds for new data and new environments, is recognised and modelled.

System analysis and decision making

From a statistical point of view, it would, of course, be preferable to testempirically such assumptions in stead of boldly implementing them in the model.

But in real-life decision making, we usually do not have large numbers of data that are representative of the concrete decisional setting of interest at our disposal.

System analysis and decision making

For instance, even for large epidemiological trials in medicine, it often remains unclear whether the resulting databases allow good generalisation to the situation in a particular hospital (due to special properties of local patients, insufficient standardisation of measurements and diagnostic procedures, etc.).

System analysis and decision making

The fact that cue interactions can exist, and that they can be covered only by fully branched tree substructures, does not imply that they must exist; it says nothing about the frequency of their occurrence.

Depending on the kind of the decision problem, there may be cases where we can make a reasonable guess about existing interactions on the substantial grounds four knowledge of the problem domain. This may, for instance, be the case for interaction effects of drugs in medical treatment.