Mathematics in Finance

Numerical solution of free boundary problems: pricing of American options

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Contents

- American options
- The obstacle problem
- Discretisation methods
- Matlab results
- Recent insights and developments

1. American options

- American options can be executed any time before expiry date, as opposed to European options that can only be exercised at expiry date
- We will derive a partial differential inequality from which a fair price for an American option can be calculated.

Bounds for prices (no dividends)

For European options:

$$(S_t - Ke^{-r(T-t)})^+ \leq C_E(S_t,t) \leq S_t$$

$$(Ke^{-r(T-t)} - S_t)^+ \leq P_E(S_t,t) \leq Ke^{-r(T-t)}$$

Reminder: put-call parity

$$S_t + P_E(S_t,t) - C_E(S_t,t) = Ke^{-r(T-t)}$$

For American options:

$$C_A(S_t,t) = C_E(S_t,t)$$

$$Ke^{-r(T-t)} \leq S_t + P_A(S_t,t) - C_A(S_t,t) \leq K$$

$$(Ke^{-r(T-t)} - S_t)^+ \leq P_A(S_t,t) \leq K$$

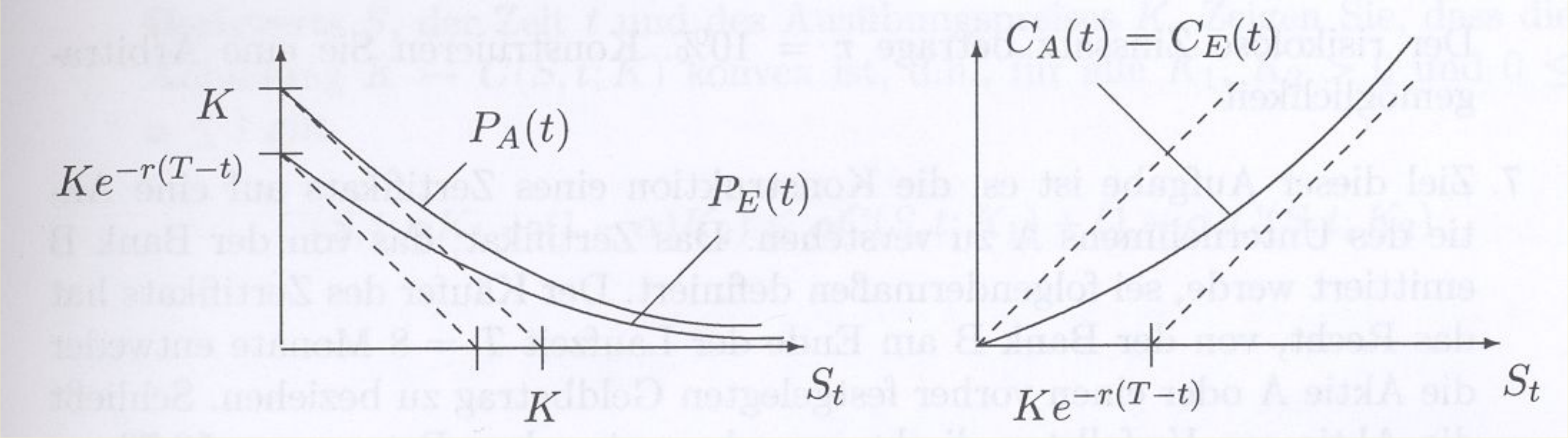
Why is $C_A(S_t,t)=C_E(S_t,t)$?

- Suppose we exercise the American call at time t<T
- Then we obtain S_t-K
- However, $C_A(S_t,t)\geq S_t-Ke^{-r(T-t)}>S_t-K$
- Hence, it is better to sell the option than to exercise it
- Consequently, the premature exercising is not optimal

What about put options?

- For put options, a similar reasoning shows that it may be advantageous to exercise at a time $t<T$
- This is due to the greater flexibility of American options

Comparison European-American options



American options are more expensive than European options

An optimum time for exercising…. (1)

Statement: There is S_f such that premature exercising is worthwhile for $S<S_f$, but not for $S>S_f$.

Proof: Let $\pi = P + S$ be a portfolio. As soon as $P = (K-S)^+ = K - S$, the option can be exercised since we can invest the amount $\pi = (K-S) + S = K$ at interest rate r. For $P > (K-S)^+$ it is not worthwhile, since the value of the portfolio before exercising is $\pi = P + S > (K-S)^+ + S \geq K$, but after exercising is equal to K.

An optimum time for exercising…. (2)

The value S_f depends on time, and it is termed the free boundary value. We have

$$P_A(S,t) = (K-S)^+ = K-S \qquad S \le S_f(t)$$

$$P_A(S,t) > (K-S)^+ \qquad S > S_f(t)$$

This free boundary value is unknown, and must be determined in addition to the option price! Therefore, we have a free boundary value problem that must be solved.

Derivation of equation and BC's (1)

- For S up to S_f the price of the put option is known
- For larger S, the put option satisfies the Black-Scholes equation since, in this case, we keep the option which can then be valued as a European option
- For S>>K, value is negligible: $P_A(S,t) \to_{S\to\infty} 0$
- Also, we must have: $P_A(S_f(t),t) = K - S_f(t)$
- Not sufficient, since we must also find S_f

Derivation of equation and BC's (2)

As extra condition, we require that

$$S \rightarrow \partial P_A(S,t)/\partial S$$

is continuous at $S=S_f(t)$. Since, for $S<S_f(t)$,

$$\partial P_A(S,t)/\partial S = \partial(K-S)/\partial S = -1$$

this can also be written in the form:

$$\boxed{\frac{\partial P_A}{\partial S}(S_f(t),t) = -1}$$

Summary of equation and BC's

The value of an American put option can be determined by solving

$$S \leq S_f(t): \qquad P_A(S,t) = K - S$$

$$S > S_f(t): \qquad \frac{\partial P_A}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_A}{\partial S^2} + (r - D_0)\frac{\partial P_A}{\partial S} - rP = 0$$

with the endpoint condition $P_A(S,T) = (K-S)^+$ and the boundary conditions:

$$\lim_{S\to\infty} P_A(S,t) = 0$$

$$P_A(S_f(t),t) = K - S_f(t)$$

$$\frac{\partial P_A}{\partial S}(S_f(t),t) = -1$$

How to solve?

- Free boundary problems can be rewritten in the form of a linear complimentarity problem, and also in alternative equivalent formulations
- These can be solved by numerical methods
- To illustrate the alternatives and the numerical solution techniques, we will give an example

2. The obstacle problem

Consider a rope:

- fixed at endpoints –1 and 1
- to be spanned over an object (given by f(x))
- with minimum length

If $f > 0, f'' < 0, x \in (a,b), f(-1), f(1) < 0$ we must find u such that:

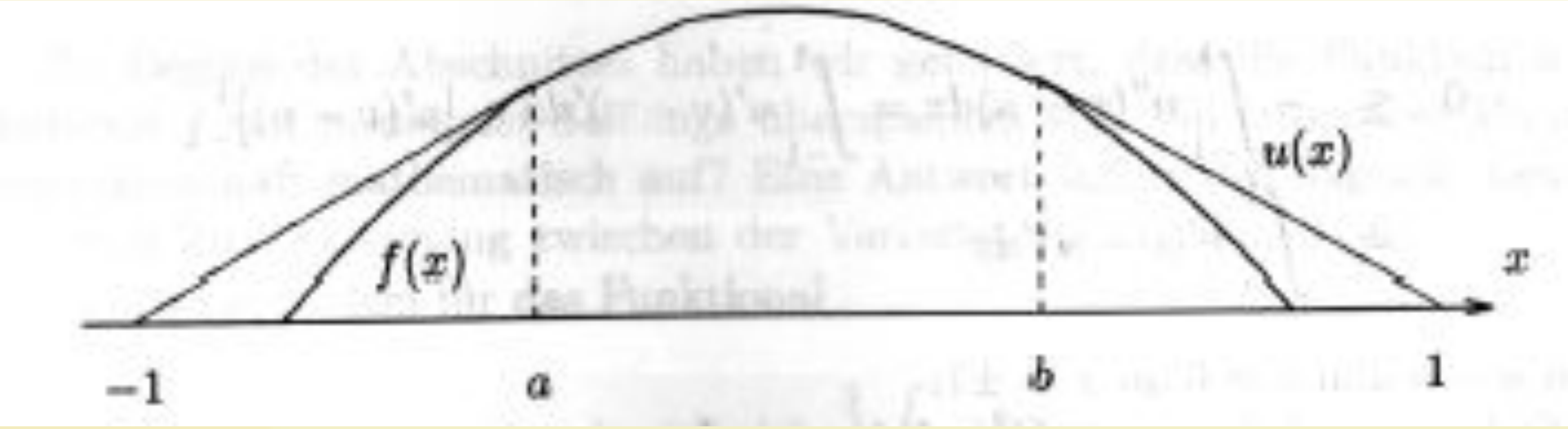
$$u \in C^1(-1,1), u(-1) = u(1) = 0$$

$$u(x) \geq f(x), x \in (-1,1)$$

$$u''(x) = 0, x \in (-1,a) \cup (b,1)$$

$$u(x) = f(x), x \in (a,b)$$

The boundaries a,b are not given, but implicitly defined.


u(x)
f(x)
x
−1
a
b
1

The linear complimentarity problem

We rewrite the above properties as follows:

$$u(x) > f(x), u''(x) = 0, x \in (-1, a)$$
$$u''(x) = f''(x) < 0, x \in (a, b)$$
$$u(x) > f(x), u''(x) = 0, x \in (b, 1)$$

and hence:

$$u(x) > f(x) \Rightarrow u''(x) = 0$$
$$u(x) = f(x) \Rightarrow u''(x) < 0$$

So we can define it as LCP:

$$u \in C^1(-1,1), u(-1) = u(1) = 0$$
$$-u'' \geq 0, u - f \geq 0, u''.(u - f) = 0, x \in (-1,1)$$

Note: free Boundaries not in formulation anymore

Formulation without second derivatives

Lemma 1: Define

$$\kappa = \{v \in C^0(-1,1) : v(-1) = v(1) = 0, v \ge f, v \in C^1_{pcw}\}$$

Then finding a solution of the LCP is equivalent to finding a solution $u \in C^2(-1,1)$ of

$$\int_{-1}^{1} u'(v-u)'\,dx \ge 0, \forall v \in \kappa$$

What about minimum length?

The latter is again equal to the following problem:

Find $u \in \kappa$ *with the property* $J(u) = \min_{v \in \kappa} J(v)$

where

$$J(v) = \frac{1}{2}\int_{-1}^{1}(v')^2 dx$$

Summarizing so far

The obstacle problem can be formulated

- As a free boundary problem
- As a linear complimentarity problem
- As a variational inequality
- As a minimization problem

We will now see how the obstacle problem can be solved numerically.

3. Discretisation methods

Finite difference method (1)

If we choose to solve the LCP, we can use the FD method. Replacing the second derivative by central differences on a uniform grid, we find the following discrete problem, to be solved $w=(w_1,\ldots,w_{N-1})$:

$$(w-f)^T Gw = 0$$

$$Gw \geq 0$$

$$w-f \geq 0$$

Here,

$$G = diag(-1,2,-1)$$

Finite difference method (2)

Alternatively, solve $\min\{Gw, w - f\} = 0$

This is equivalent to solving

$$\min\{w - D^{-1}(Lw + Uw), w - f\} = 0$$

Or:

$$w = \max\{D^{-1}(Lw + Uw), f\}$$

Finite difference method (3)

We can use the projection SOR method to solve this problem iteratively: for i=1,…,N-1:

$$z_i^{(k)} = a_{ii}^{-1}(Lw^{(k+1)} + Uw^{(k)})_i$$

$$w_i^{(k+1)} = \max\{w_i^{(k)} + \omega(z_i^{(k)} - w_i^{(k)}), f_i\}$$

A theorem by Cryer proves that this sequence converges (for posdef G and 1<omega<2)

Finite element method (1)

As the basis we use the variational inequality

$$\int_{-1}^{1} u'(v-u)'\,dx \geq 0, \forall v \in \kappa$$

The basic idea is to solve this equation in a smaller space $\kappa^* \subset \kappa$. We choose simple piecewise linear functions on the same mesh as used for the FD.

Hence, we may write

$$u(x) = \sum_{i=1}^{N-1} u_i \phi_i(x), v(x) = \sum_{i=1}^{N-1} v_i \phi_i(x)$$

Finite element method (2)

These expressions can be substituted in the variational inequality. Working out the integrals (simple), we find the following discrete inequality (G as in FD):

$$u^T G(v-u) \geq 0$$

This must be solved in conjunction with the constraint that

$$u \geq f$$

Proposition:

The above FEM problem is the same as the problem generated by the FD method.

Summary: comparison of FD and FEM

Finite difference method:

$$(u-f)^T Gu = 0$$

$$Gu \geq 0$$

$$u-f \geq 0$$

Finite element method:

$$u^T G(v-u) \geq 0, \forall v \in \kappa^*$$

$$v-f \geq 0$$

$$u-f \geq 0$$

4. Implementation in Matlab

Back to American options

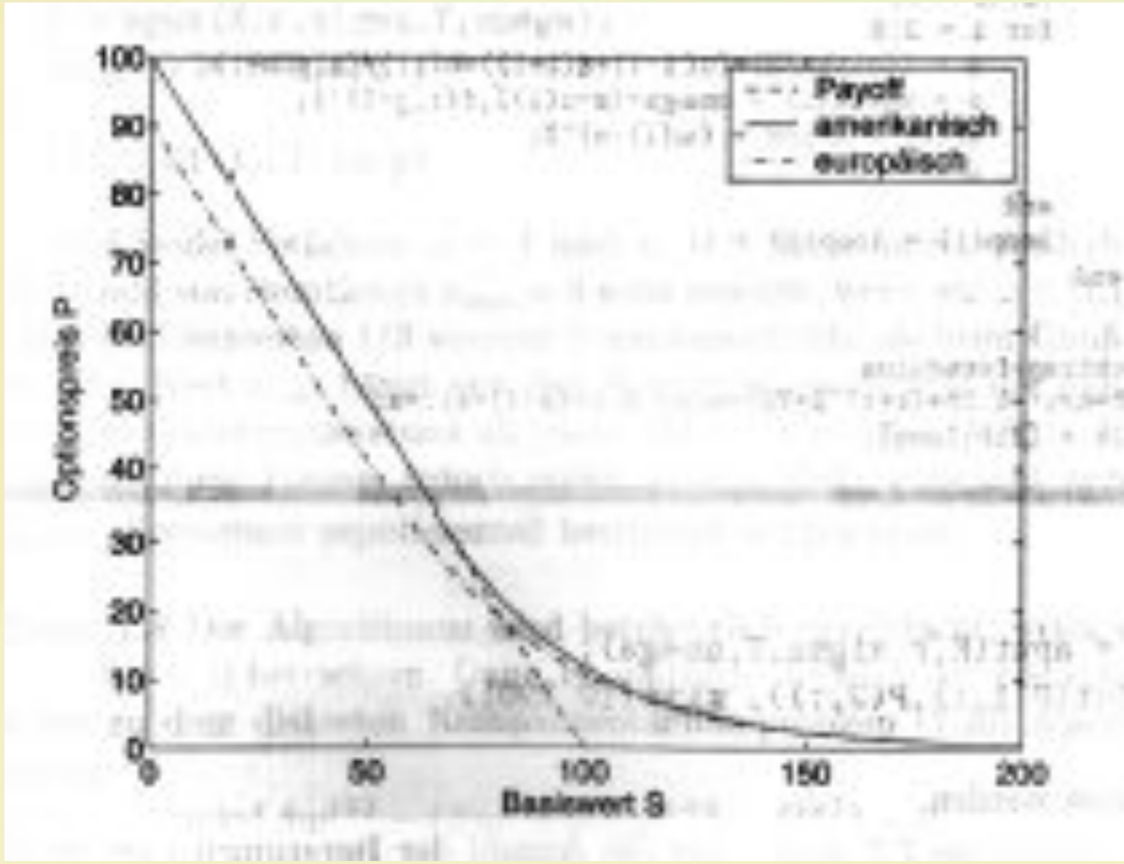
The problem for American options is very similar to the obstacle problem, so the treatment is also similar. First, the problem is formulated as a linear complimentarity problem, containing a Black-Scholes inequality, which can be transformed into the following system (cf. the variational form!):

$$(V-\Lambda(S)).(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D_0)S\frac{\partial V}{\partial S}-rV)=0$$

$$-(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D_0)S\frac{\partial V}{\partial S}-rV)\geq 0$$

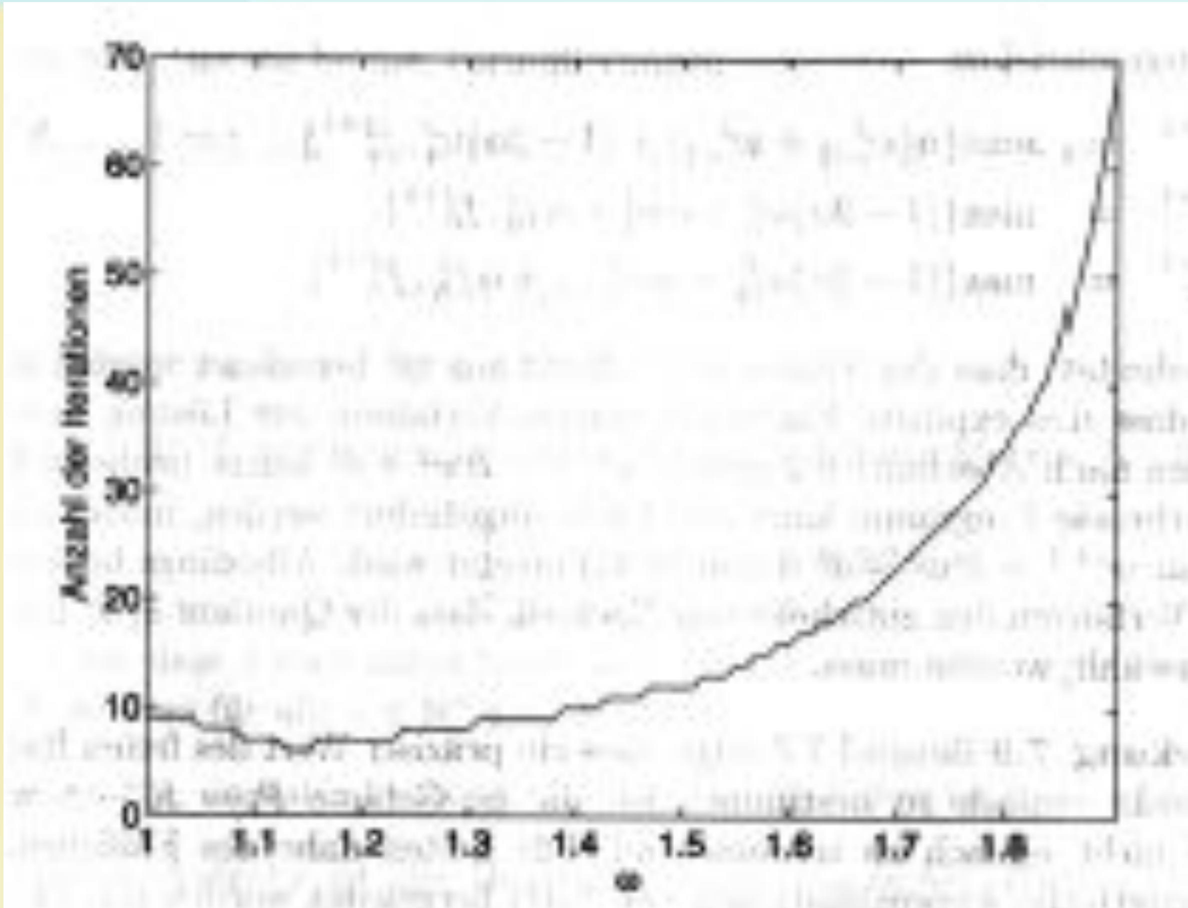
$$V-\Lambda(S)\geq 0$$

$$\Lambda(S)=(K-S)^+$$



Result of Matlab calculation using projection SOR

K=100, r=0.1, sigma=0.4, T=1



Number of iterations in projection SOR method
Depending on the overrelaxation parameter omega

5. Recent insights and developments

Historical account

- First widely-used methods using FD by Brennan and Schwartz (1977) and Cox et al. 1979)
- Wilmott, Dewynne and Howison (1993) introduced implicit FD methods for solving PDE's, by solving an LCP at each step using the projected SOR method of Cryer (1971)
- Huang and Pang (1998) gave a nice survey of state-of-the-art numerical methods for solving LCP's. Unfortunately, they assume a regular FD grid

Recent work (1)

- **Some people concentrate on Monte Carlo methods to evaluate the discounted integrals of the payoff function**
- **More popular are the QMC methods that are more efficient (Niederreiter, 1992)**

Recent insight: PDE methods may be preferable to MC methods for American option pricing:

- **PDE methods typically admit Taylor series analyses for European problems, whereas simulation-based methods admit less optimistic probabilistic error analyses**
- **The number of tuning parameters that must be used in PDE methods is much smaller that that required for simulation-based techniques that have been suggested for American option pricing**

Recent work (2)

In

S. Berridge

"Irregular Grid Methods for Pricing High-Dimensional American Options"

(Tilburg University, 2004)

an account is given of several methods based on the use of irregular grids.