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Physics 1

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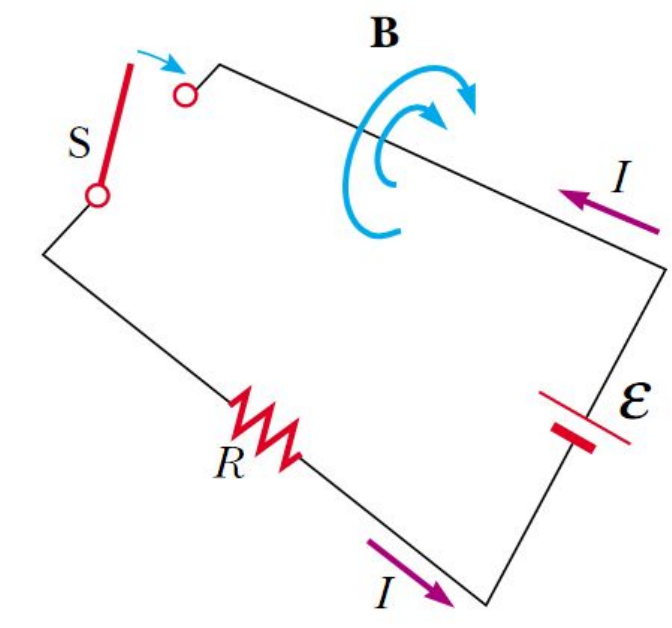
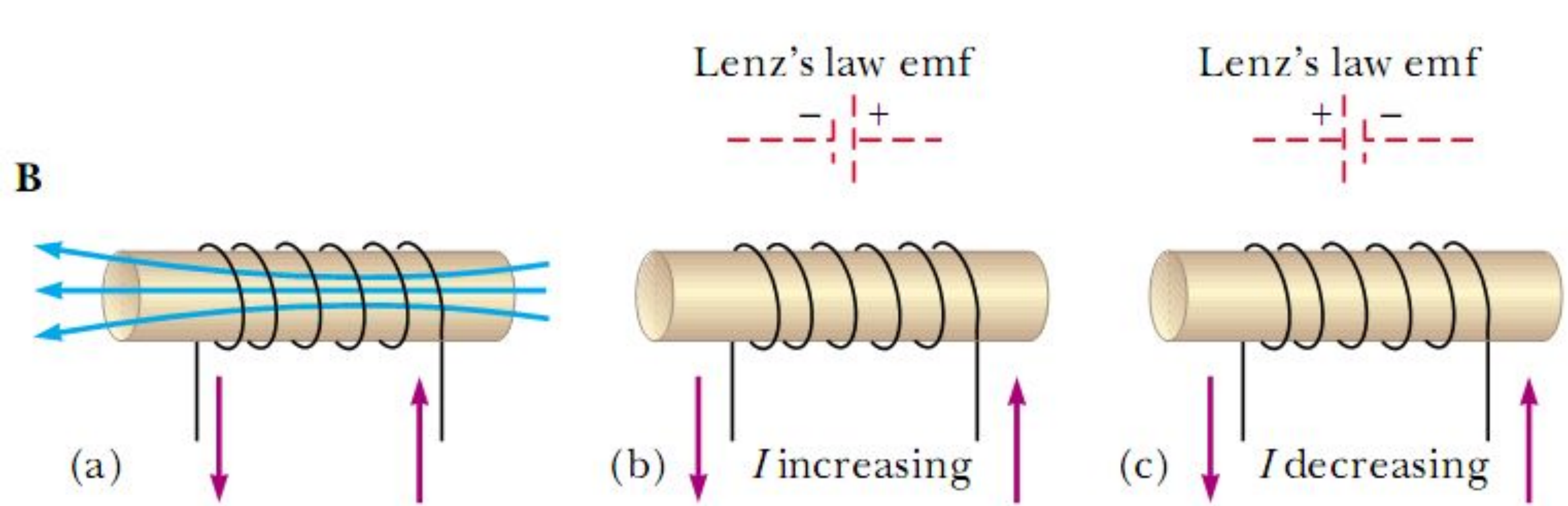
Lecture 14

- Inductance
- Self-inductance
- RL Circuits
- Energy in a Magnetic Field
- Mutual inductance
- LC circuit – harmonic oscillations
- RLC circuit – damped harmonic oscillations

Self-inductance



When the switch is thrown to its closed position, the current does not immediately jump from zero to its maximum value ε/R. As the current increases with time, the magnetic flux through the circuit loop due to this current also increases with time. This increasing flux creates an induced emf in the circuit. The direction of the induced emf is such that it would cause an induced current in the loop), which would establish a magnetic field opposing the change in the original magnetic field. Thus, the direction of the induced emf is opposite the direction of the emf of the battery; this results in a gradual rather than instantaneous increase in the current to its final equilibrium value. Because of the direction of the induced emf, it is also called a back emf. This effect is called **self-induction** because the changing flux through the circuit and the resultant induced emf arise from the circuit itself. The emf ε_L set up in this case is called a self-induced emf.



(a) A current in the coil produces a magnetic field directed to the left.

(b) If the current increases, the increasing magnetic flux creates an induced emf in the coil having the polarity shown by the dashed battery.

(c) The polarity of the induced emf reverses if the current decreases.

Self-induced emf

From Faraday's law follows that the induced emf is equal to the negative of the time rate of change of the magnetic flux. The magnetic flux is proportional to the magnetic field due to the current, which in turn is proportional to the current in the circuit. Therefore, a self-induced emf is always proportional to the time rate of change of the current:

$$\mathcal{E}_L = -L\frac{dI}{dt}$$

L is a proportionality constant—called the inductance of the coil—that depends on the geometry of the coil and other physical characteristics.

- From last expression it follows that

$$L = -\frac{\mathcal{E}_L}{dI/dt}$$

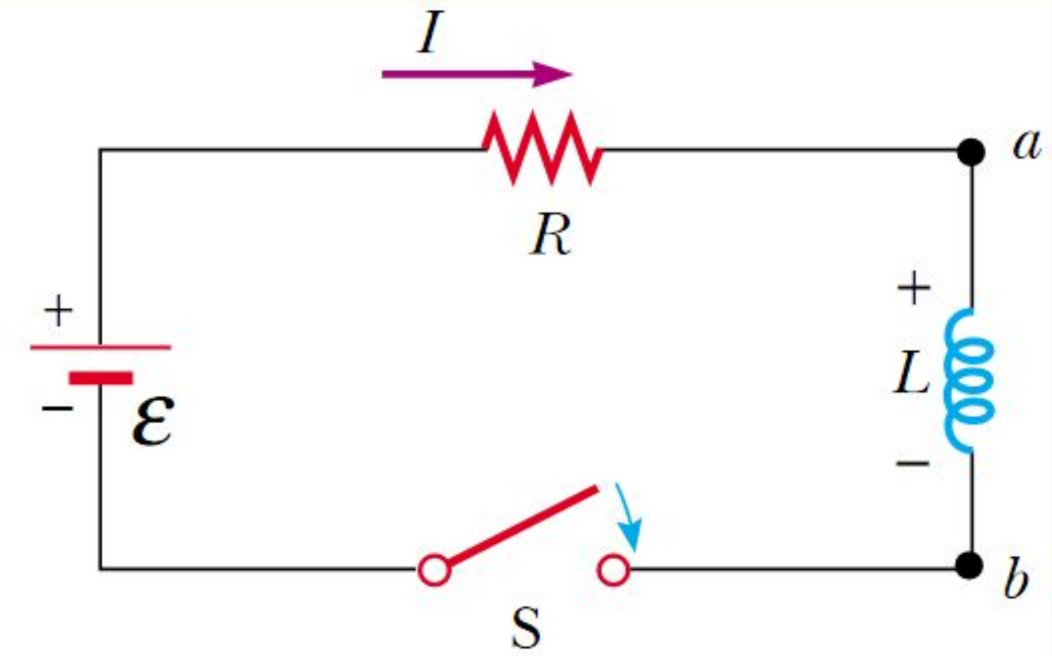
- So **inductance** is a measure of the opposition to a change in current.

Ideal Solenoid Inductance

Combining the last expression with Faraday's law, $\varepsilon L = -N\, d\Phi_B/dt$, we see that the inductance of a closely spaced coil of N turns (a toroid or an ideal solenoid) carrying a current I and containing N turns is

$$L = \frac{N\Phi_B}{I}$$

Series RL Circuit



An inductor in a circuit opposes changes in the current in that circuit:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

$$x + \frac{L}{R}\frac{dx}{dt} = 0$$

$$\frac{dx}{x} = -\frac{R}{L}\,dt$$

$$\int_{x_0}^{x} \frac{dx}{x} = -\frac{R}{L}\int_0^t dt$$

$$\ln\frac{x}{x_0} = -\frac{R}{L}\,t$$

- Taking the antilogarithm of the last result:

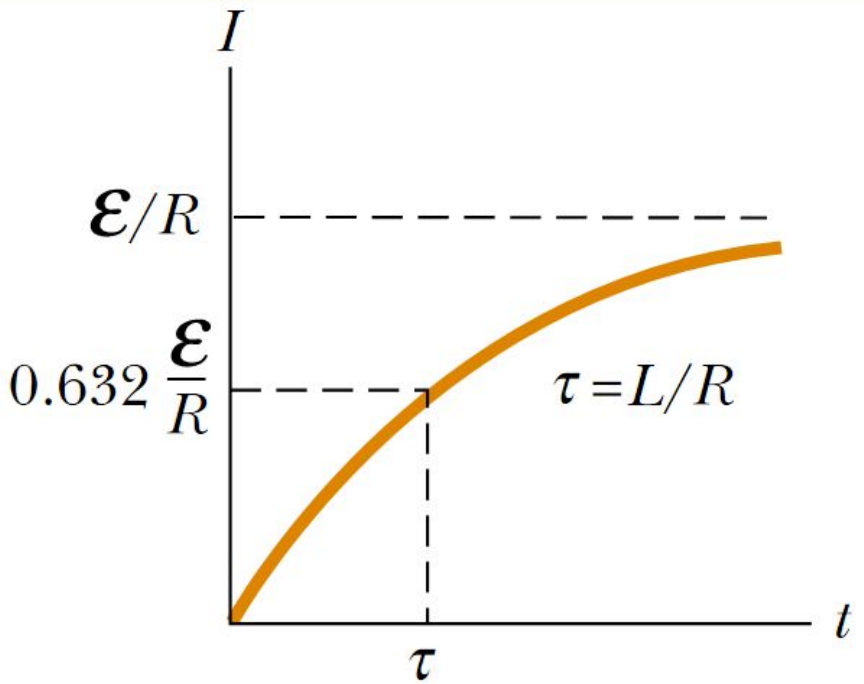
$$x = x_0 e^{-Rt/L}$$

- Because I = 0 at t = 0, we note from the definition of x that $x_0 = \varepsilon/R$. Hence, this last expression is equivalent to

$$\frac{\varepsilon}{R} - I = \frac{\varepsilon}{R} e^{-Rt/L}$$

$$I = \frac{\varepsilon}{R} (1 - e^{-Rt/L})$$

So the current gradually approaches its maximum:



The time constant τ is the time interval required for I to reach 0.632 ($1-e^{-1}$) of its maximum value.

Energy in an Inductor

Multiplying by I the expression for RL–circuit we obtain:

$$I\mathcal{E} = I^2R + LI\frac{dI}{dt}$$

So here $I\varepsilon$ is the power output of the battery, I^2R is the power dissipated on the resistor, then $LIdI/dt$ is the power delivering to the inductor. Let's U denote as the energy stored in the inductor, then:

$$\frac{dU}{dt} = LI\frac{dI}{dt}$$

- After integration of the last formula:

$$U = \frac{1}{2} L I^2$$

- L is the inductance of the inductor,
- I is the current in the inductor,
- U is the energy stored in the magnetic field of the inductor.

Magnetic Field Energy Density

- Inductance for solenoid is:

$$L = \mu_0 n^2 A\ell$$

- The magnetic field of a solenoid is:

$$B = \mu_0 n I$$

- Then: $U = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 n^2 A\ell \left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0} A\ell$

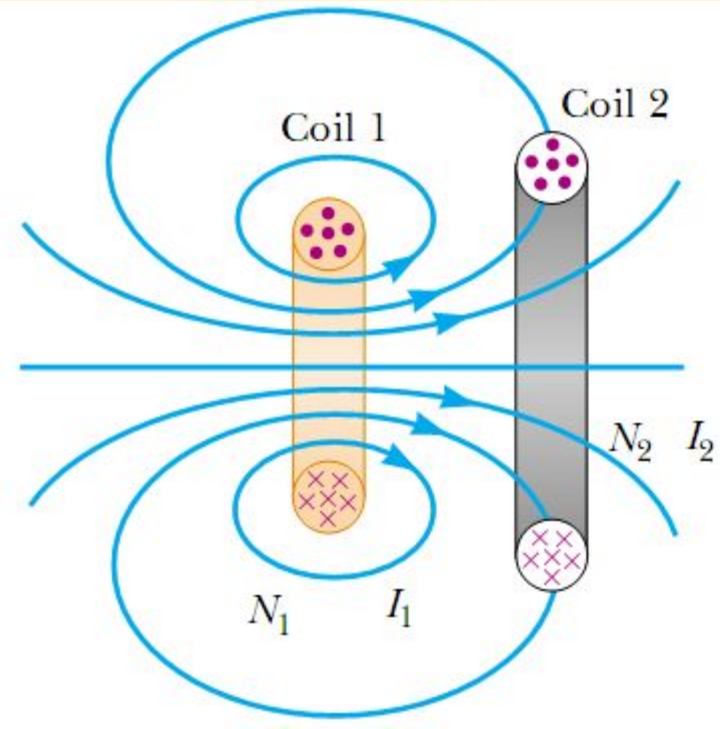
- Al is the volume of the solenoid, then the energy density of the magnetic field is:

$$u_B = \frac{U}{A\ell} = \frac{B^2}{2\mu_0}$$

$$u_B = \frac{B^2}{2\mu_0}$$

- u_B is the energy density of the magnetic field
- B is the magnetic field vector
- μ_0 is the free space permeability for the magnetic field, a constant.
- Though this formula was obtained for solenoid, it's valid for any region of space where a magnetic field exists.

Mutual Inductance



A cross-sectional view of two adjacent coils. The current I_1 in coil 1, which has N_1 turns, creates a magnetic field. Some of the magnetic field lines pass through coil 2, which has N_2 turns. The magnetic flux caused by the current in coil 1 and passing through coil 2 is represented by F_{12}. The mutual inductance M_{12} of coil 2 with respect to coil 1 is:

$$M_{12} \equiv \frac{N_2 \Phi_{12}}{I_1}$$

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- The emf induced by coil 1 in coil 2 is:

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{12}}{dt} = -N_2 \frac{d}{dt}\left(\frac{M_{12} I_1}{N_2}\right) = -M_{12} \frac{dI_1}{dt}$$

- The preceding discussion can be repeated to show that there is a mutual inductance M21. The emf induced by coil 1 in coil 2 is:

$$\mathcal{E}_1 = -M_{21} \frac{dI_2}{dt}$$

- **In mutual induction, the emf induced in one coil is always proportional to the rate at which the current in the other coil is changing.**

Although the proportionality constants M_{12} and M_{21} have been obtained separately, it can be shown that they are equal. Thus, with $M_{12} = M_{21} = M$, the expressions for induced emf takes the form:

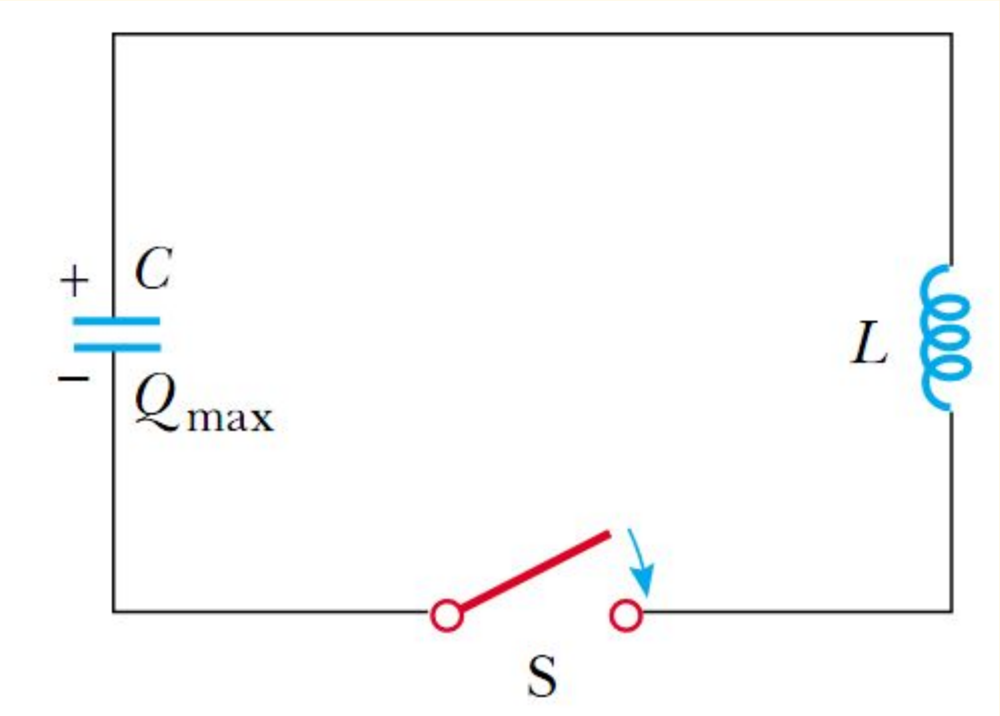
$$\varepsilon_1 = -M \frac{dI_2}{dt}$$

$$\varepsilon_2 = -M \frac{dI_1}{dt}$$

These two expression are similar to that for the self-induced emf: $\varepsilon = - L(dI/dt)$.

The unit of mutual inductance is the henry.

LC Circuit Oscillations



If the capacitor is initially charged and the switch is then closed, we find that both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values.

We assume:

– the resistance of the circuit is zero, then no energy is dissipated,

– energy is not radiated away from the circuit.

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2} L I^2$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2} L I^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0$$

$$\frac{Q}{C} + L\frac{d^2 Q}{dt^2} = 0$$

$$\frac{d^2 Q}{dt^2} = -\frac{1}{LC} Q$$

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- The solution for the equation is:

$$Q = Q_{max}\cos(\omega t + \phi)$$

$$\omega = \frac{1}{\sqrt{LC}}$$

- The angular frequency of the oscillations depends solely on the inductance and capacitance of the circuit. This is the **natural frequency** (частота собственных колебаний) of oscillation of the LC circuit.

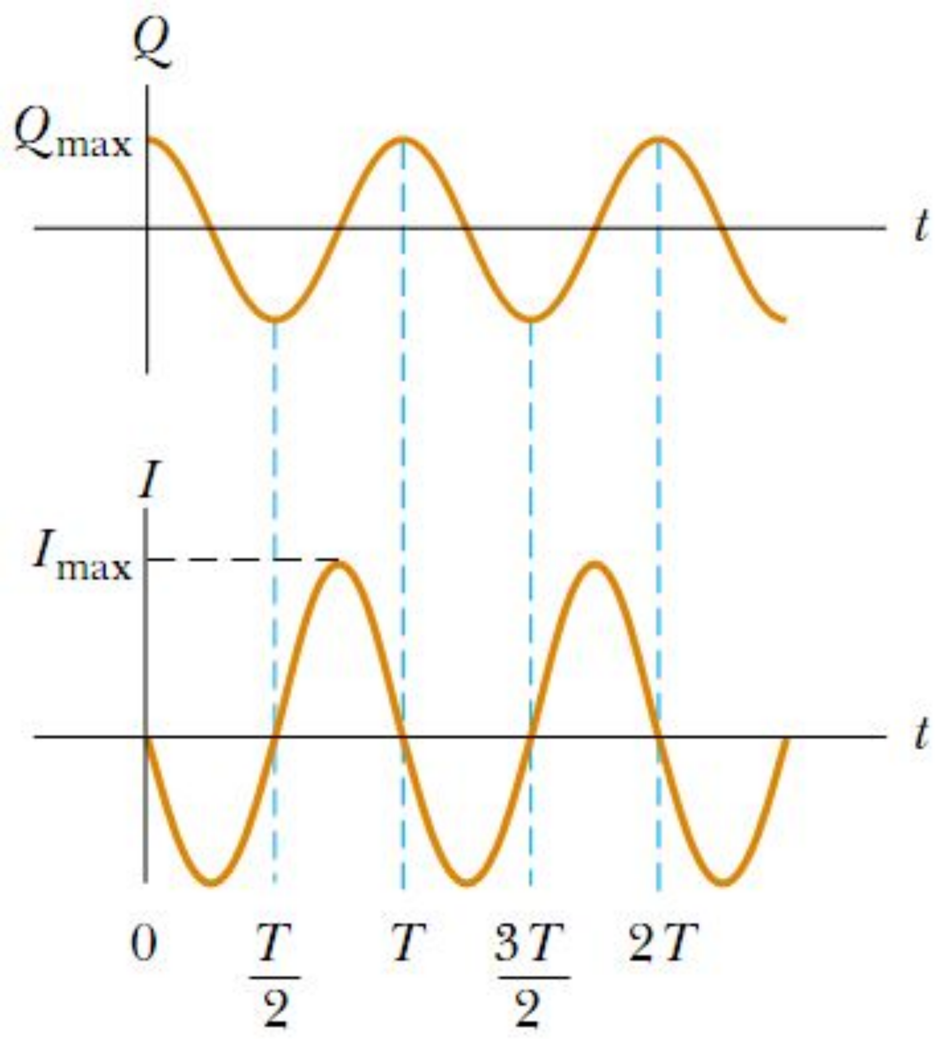
- Then the current is:

$$I = \frac{dQ}{dt} = -\omega Q_{\max} \sin(\omega t + \phi)$$

- Choosing the initial conditions: at $t = 0$, $I = 0$ and $Q = Q_{max}$ we determine that $\varphi=0$.
- Eventually, the charge in the capacitor and the current in the inductor are:

$$Q = Q_{\max} \cos \omega t$$

$$I = -\omega Q_{\max} \sin \omega t = -I_{\max} \sin \omega t$$

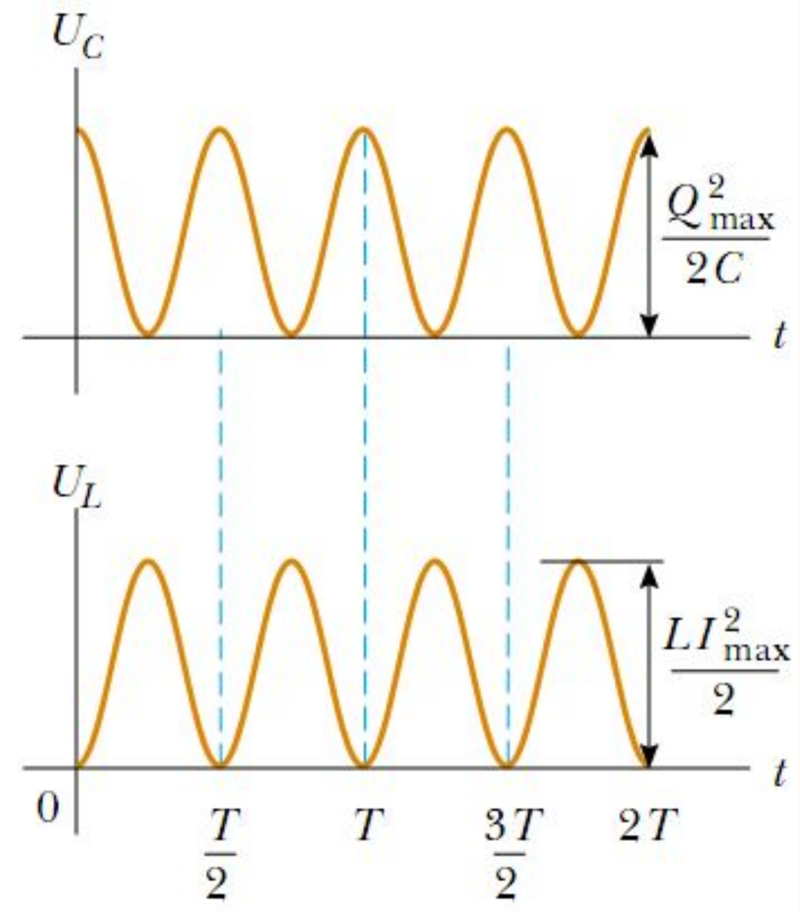


Graph of charge versus time

and

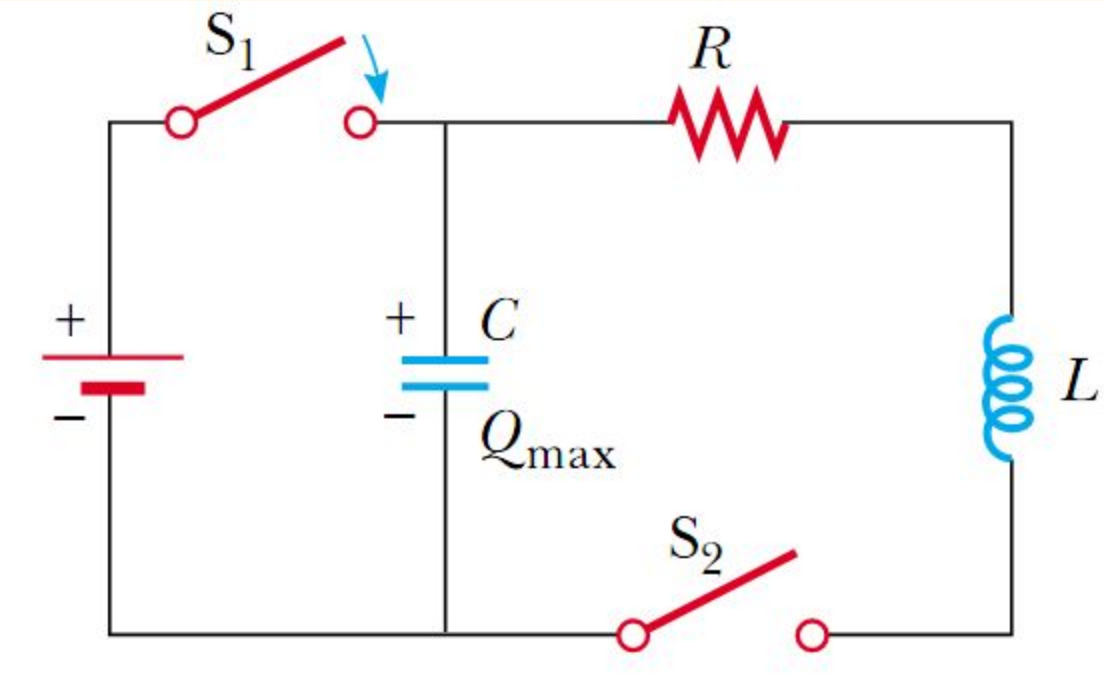
Graph of current versus time for a resistanceless, nonradiating LC circuit.

NOTE: Q and I are 90° out of phase with each other.



- Plots of U_C versus t and U_L versus t for a resistanceless, nonradiating LC circuit.
- The sum of the two curves is a constant and equal to the total energy stored in the circuit.

RLC circuit



A series RLC circuit. Switch S_1 is closed and the capacitor is charged. S_1 is then opened and, at t = 0, switch S_2 is closed.

- Energy is dissipated on the resistor:

$$\frac{dU}{dt} = -I^2R$$

- Using the equation for dU/dt in the LC-circuit (slide 2):

$$LI\frac{dI}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -I^2R$$

- Using that *I=dQ/dt*:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

- The RLC circuit is analogous to the damped harmonic oscillator, where R is damping coefficient.

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

- Here b is damping coefficient. When b=0, we have pure harmonic oscillations.
- Solution is:

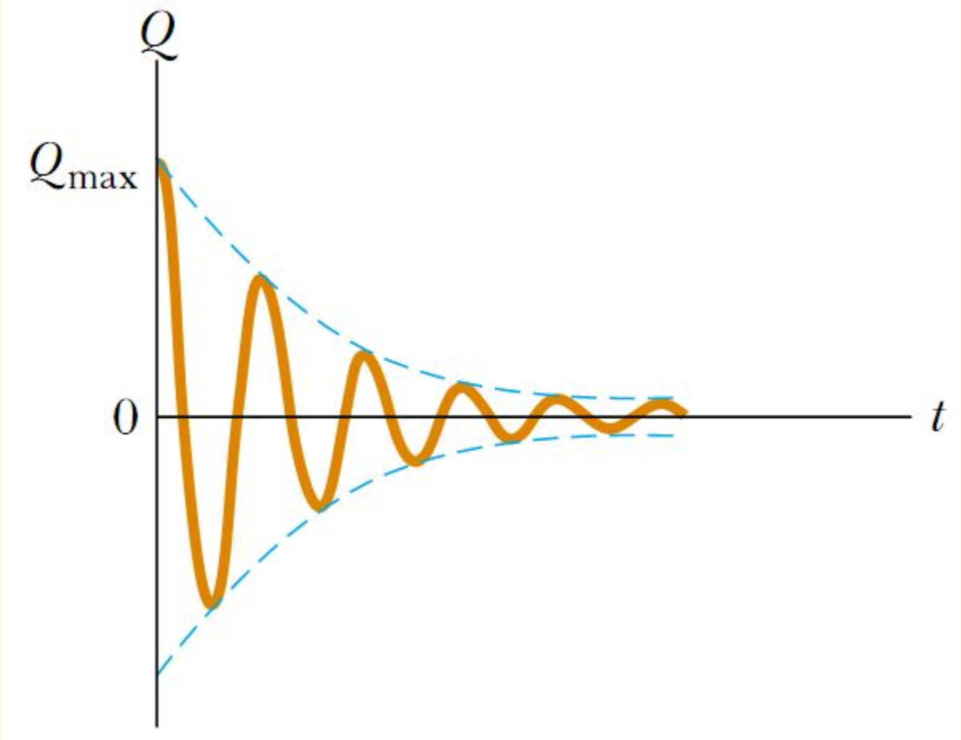
$$Q = Q_{\max} e^{-Rt/2L} \cos \omega_d t$$

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L} \right)^2 \right]^{1/2}$$

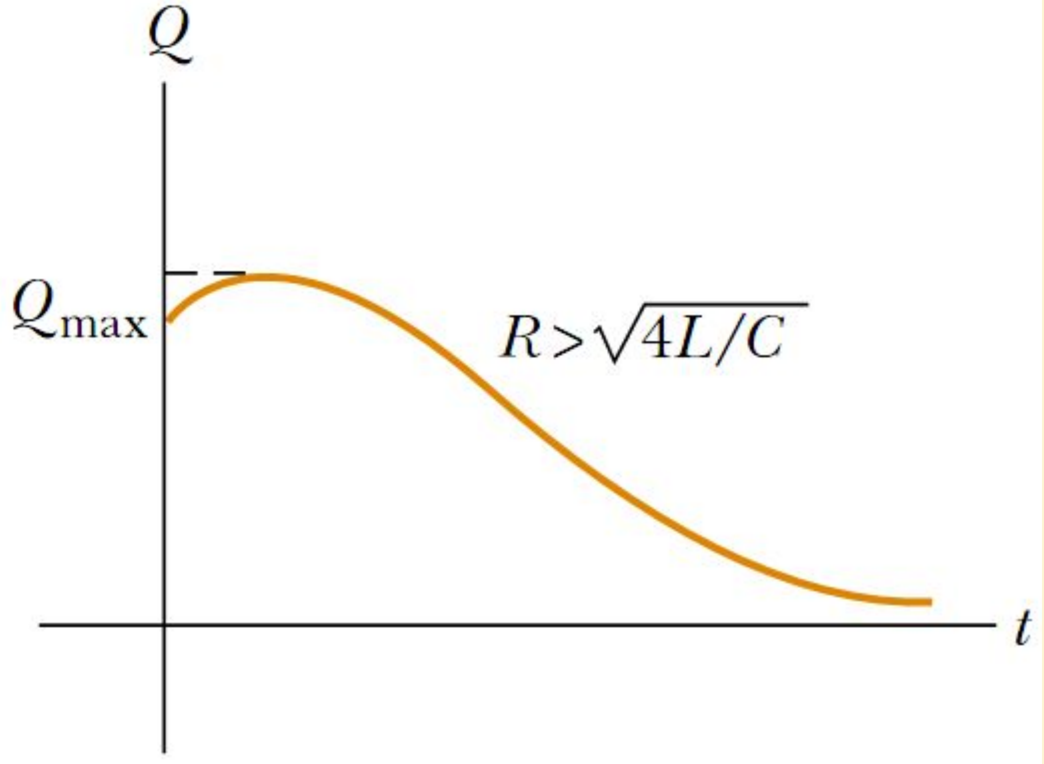
- R_C is the critical resistance.

$$R_c = \sqrt{4L/C}$$

- When $R<R_C$ oscillations are damped harmonic.
- When $R>R_C$ oscillations are damped unharmonic.



$$R < \sqrt{4L/C}$$


Q
Q_{max}
$R > \sqrt{4L/C}$
t

Units in Si

- Inductance L H (henry): 1H=V*s/A
- Mutual Inductance M H (henry): 1H=V*s/A
- Energy density u J/m^3