Chapter 12: Kinematics of a Particle

Section 12.1: Introduction

Learning objective

Be able to find the kinematic quantities (position, displacement, velocity, and acceleration) of a particle traveling along a straight path.

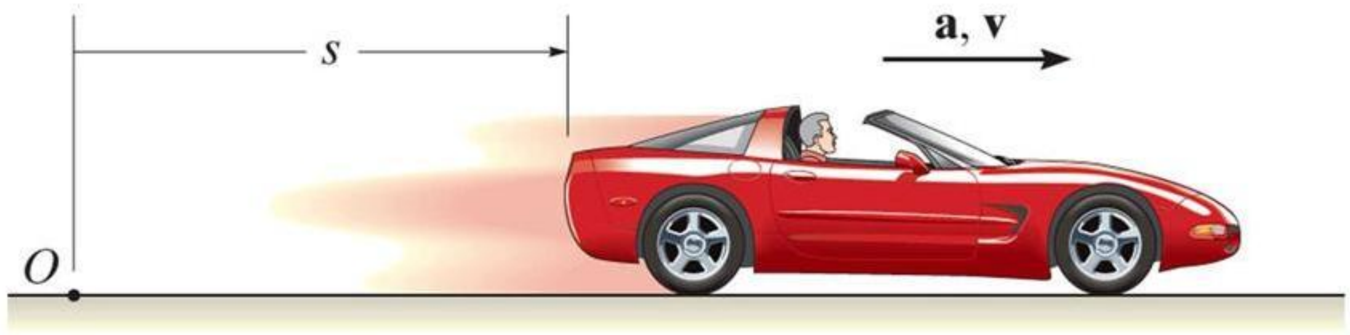
Applications

The motion of large objects, such as rockets, airplanes, or cars, can often be analyzed as if they were particles.

Why?

If we measure the altitude of this rocket as a function of time, how can we determine its velocity and acceleration?

Applications



A sports car travels along a straight road.

Can we treat the car as a particle?

If the car accelerates at a constant rate, how can we determine its position and velocity at some instant?

An Overview of Mechanics

Mechanics: The study of how bodies react to forces acting on them.

Statics: The study of bodies in equilibrium.

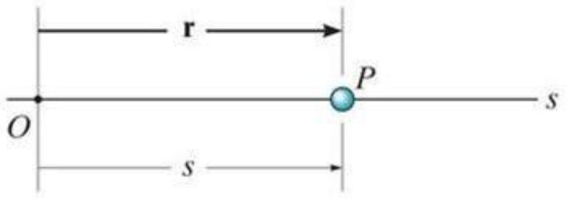
Dynamics:

1. **Kinematics** – concerned with the geometric aspects of motion
2. **Kinetics –** concerned with the forces causing the motion

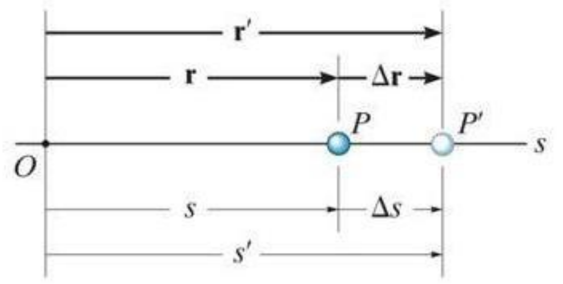
Chapter 12:
Kinematics of a Particle

Section 12.2: Rectilinear Kinematics: Continuous Motion

Continuous Motion



Position



Displacement

A particle travels along a straight-line path defined by the coordinate axis s.

The position of the particle at any instant, relative to the origin, O, is defined by the position vector ***r***, or the scalar s. Scalar s can be positive or negative. The typical unit is meter (m).

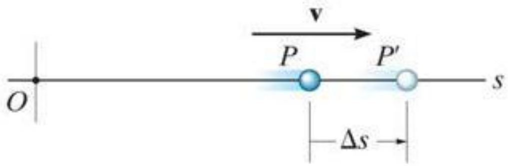
The displacement of the particle is defined as its change in position.

Vector form: $\Delta \mathbf{r} = \mathbf{r}' - \mathbf{r}$ Scalar form: $\Delta s = s' - s$

The total distance traveled by the particle, s_T, is a positive scalar that represents the total length of the path over which the particle travels.

Velocity

Velocity is a measure of the rate of change in the position of a particle. It is a vector quantity (it has both magnitude and direction). The magnitude of the velocity is called speed, with unit m/s.

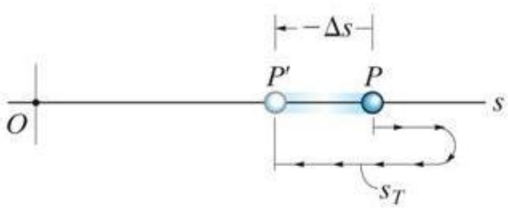


Velocity

The average velocity of a particle during a time interval Δt is

$$\boldsymbol{v}_{avg} = \boldsymbol{\Delta r} / \Delta t$$

The instantaneous velocity is the time-derivative of position. $\boldsymbol{v} = d\boldsymbol{r}/dt$



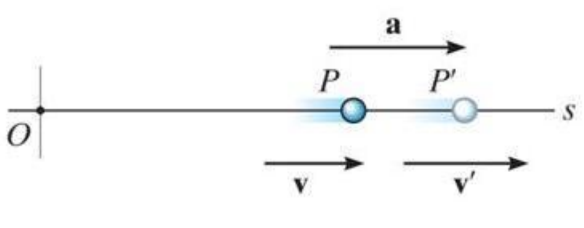
Average velocity and
Average speed

Speed is the magnitude of velocity:
v = ds/dt

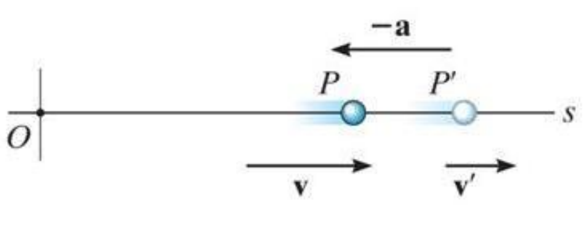
Average **speed** is the total distance traveled divided by elapsed time:
$(v_{sp})_{avg} = s_T / \Delta t$ (only used occasionally)

Acceleration

Acceleration is the rate of change in the velocity of a particle. It is a vector quantity. Typical units are m/s^2.



Acceleration



Deceleration

The instantaneous acceleration is the time derivative of velocity.

Vector form: $\mathbf{a} = d\boldsymbol{v}/dt$

Scalar form: $a = dv/dt = d^2s/dt^2$

Acceleration can be positive (speed increasing) or negative (speed decreasing).

As the text indicates, the derivative equations for velocity and acceleration can be manipulated to get: $a\ ds = v\ dv$

Summary of Kinematic Relations

- Differentiate position to get velocity and acceleration.

$$v = ds/dt\ ; \qquad a = dv/dt \quad \text{or} \quad a = v\, dv/ds$$

- Integrate acceleration for velocity and position.

Velocity:

$$\int_{v_o}^{v} dv = \int_{o}^{t} a\,dt \quad \text{or} \quad \int_{v_o}^{v} v\,dv = \int_{s_o}^{s} a\,ds$$

Position:

$$\int_{s_o}^{s} ds = \int_{o}^{t} v\,dt$$

- Note that s_o and v_o represent the initial position and velocity of the particle at $t = 0$.

Constant Acceleration

The three kinematic equations can be integrated for the special case when acceleration is constant ($a = a_c$) to obtain very useful equations. A common example of constant acceleration is gravity; i.e., a body freely falling toward earth. In this case, $a_c = g = 9.81\ m/s^2$ downward. These equations are:

$$\int_{v_o}^{v} dv = \int_{o}^{t} a_c\, dt \quad \text{yields} \quad v = v_o + a_c t$$

$$\int_{s_o}^{s} ds = \int_{o}^{t} v\, dt \quad \text{yields} \quad s = s_o + v_o t + (1/2) a_c t^2$$

$$\int_{v_o}^{v} v\, dv = \int_{s_o}^{s} a_c\, ds \quad \text{yields} \quad v^2 = (v_o)^2 + 2 a_c (s - s_o)$$

Example

Given: A particle travels along a straight line to the right with a velocity of $v = (4t - 3t^2)$ m/s where t is in seconds. Also, s = 0 when t = 0.

Find: The position and acceleration of the particle when t = 4 s.

Plan: Establish the positive coordinate, s, in the direction the particle is traveling. Since the velocity is given as a function of time, take a derivative of it to calculate the acceleration. Conversely, integrate the velocity function to calculate the position.

Solution

1) Take a derivative of the velocity to determine the acceleration.

$a = dv / dt \quad = d(4\,t - 3\,t^2) / dt = 4 - 6\,t$

$\Rightarrow \quad a = -20\ m/s^2$ (or in the $\leftarrow$ direction) when $t = 4$ s

2) Calculate the distance traveled in 4s by integrating the velocity using $s_o = 0$:

$v = ds / dt \quad \Rightarrow \quad ds = v\,dt \quad \Rightarrow \qquad \int_{s_o}^{s} ds = \int_{o}^{t} (4\,t - 3\,t^2)\,dt$

$\Rightarrow \quad s - s_o = 2\,t^2 - t^3$

$\Rightarrow \quad s - 0 = 2(4)^2 - (4)^3 \quad \Rightarrow \quad s = -32$ m (or $\leftarrow$)

Channel Setting Instructions for ***ResponseCard RF***

1. Press and release the "GO" or "CH" button.

2. While the light is flashing red and green, enter the 2 digit channel code (i.e. channel 1 = 01, channel 21 = 21).

   Channel is 11

3. After the second digit is entered, Press and release the "GO" or "CH" button. The light should flash green to confirm.

4. Press and release the "1/A" button. The light should flash amber to confirm.

Quiz

A particle moves along a horizontal path with its velocity varying with time as shown. The average acceleration of the particle is?

1. 0.4 m/s^2 $\rightarrow$
2. 0.4 m/s^2 $\leftarrow$
3. 1.6 m/s^2 $\rightarrow$
4. 1.6 m/s^2 $\leftarrow$

A particle has an initial velocity of 30 m/s to the left. If it then passes through the same location 5 seconds later with a velocity of 50 m/s to the right, the average velocity of the particle during the 5 s time interval is?

1. 10 m/s $\longrightarrow$
2. 40 m/s $\longrightarrow$
3. 16 m/s $\longrightarrow$
4. 0 m/s

Example

Given: A particle is moving along a straight line such that its velocity is defined as $v = (-4s^2)$ m/s, where s is in meters.

Find: The velocity and acceleration as functions of time if $s = 2$ m when $t = 0$.

Plan: Since the velocity is given as a function of distance, use the equation $v=ds/dt$.

1) Express the distance in terms of time.
2) Take a derivative of it to calculate the velocity and acceleration.

Solution

1) Since $v = (-4s^2)$

$$v = \frac{ds}{dt} = -4\,s^2 \;\Rightarrow\; -4\,dt = \frac{ds}{s^2}$$

Determine the distance by integrating using $s_0 = 2$.

$$\int_0^t (-4)dt = \int_2^s s^{-2} ds$$

Notice that s = 2 m when t = 0.

$$-4\,t\Big]_0^t = -\frac{1}{s}\Big]_2^s \;\Rightarrow\; -4\,t = -\left(\frac{1}{s} - \frac{1}{2}\right)$$

$$\Rightarrow\; 4\,t + \frac{1}{2} = \frac{1}{s} \;\Rightarrow\; s = \frac{2}{8t+1}$$

Solution

2) Take a derivative of distance to calculate the velocity and acceleration.

$$s = \frac{2}{8t + 1} \quad \text{m}$$

$$\Rightarrow \quad v = \frac{ds}{dt} = \frac{2 \cdot (-1) \cdot 8}{(8t + 1)^2} = -\frac{16}{(8t + 1)^2} \quad \text{m/s}$$

$$\Rightarrow \quad a = \frac{dv}{dt} = -\frac{16 \cdot (-2) \cdot 8}{(8t + 1)^3} = \frac{256}{(8t + 1)^3} \quad \text{m/s}^2$$

Quiz

A particle has an initial velocity of 3 m/s to the left at s_0 = 0 m. Determine its position when t = 3 s if the acceleration is 2 m/s^2 to the right.

1. 0.0 m
2. 6.0 m ←
3. 18.0 m →
4. 9.0 m →

A particle is moving with an initial velocity of v = 12 m/s and constant acceleration of 3.78 m/s^2 in the same direction as the velocity. Determine the distance the particle has traveled when the velocity reaches 30 m/s.

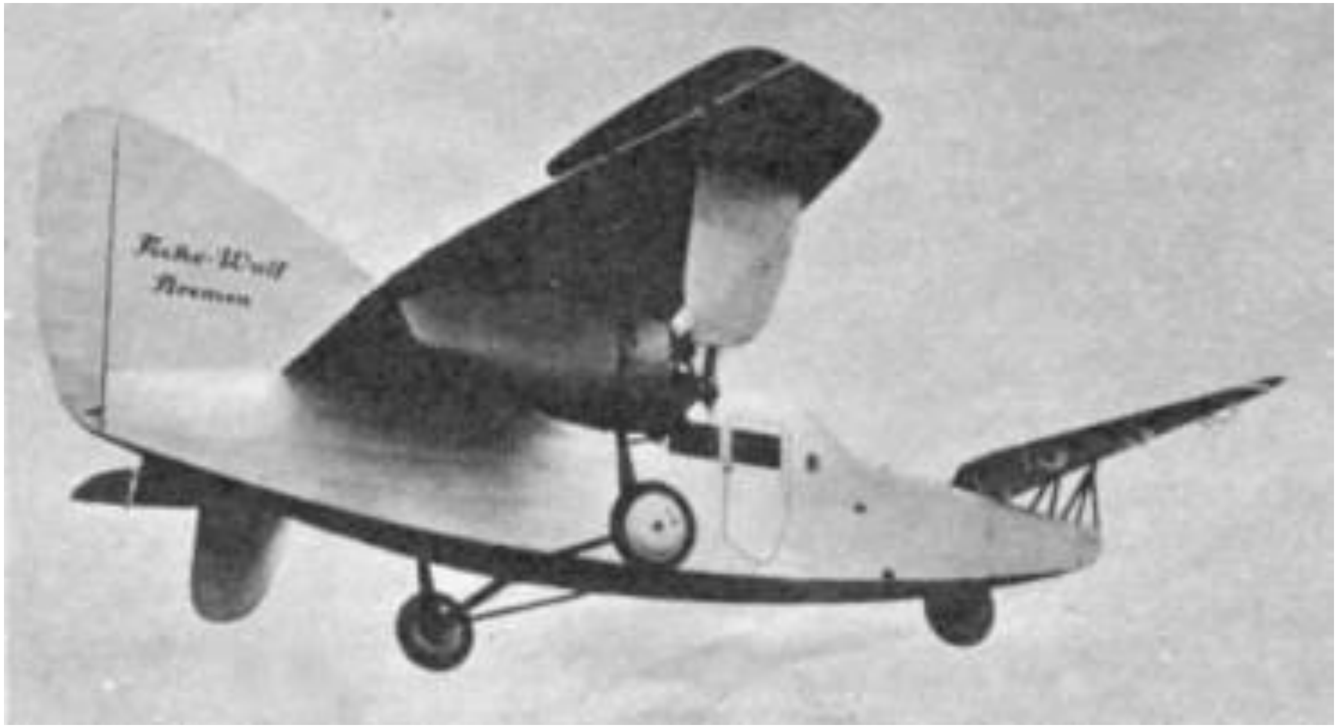
1. 50 m
2. 100 m
3. 150 m
4. 200 m

Ugly aircraft competition

Scale of Ugliness

1 = most beautiful aircraft ever built
2 = extremely beautiful aircraft
3 = very beautiful
4 = pretty beautiful
5 = beautiful
6 = ugly
7 = pretty ugly
8 = very ugly
9 = extremely ugly aircraft
10 = most ugly aircraft ever built

Focke Wulf 19a Ente (1927)



1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10

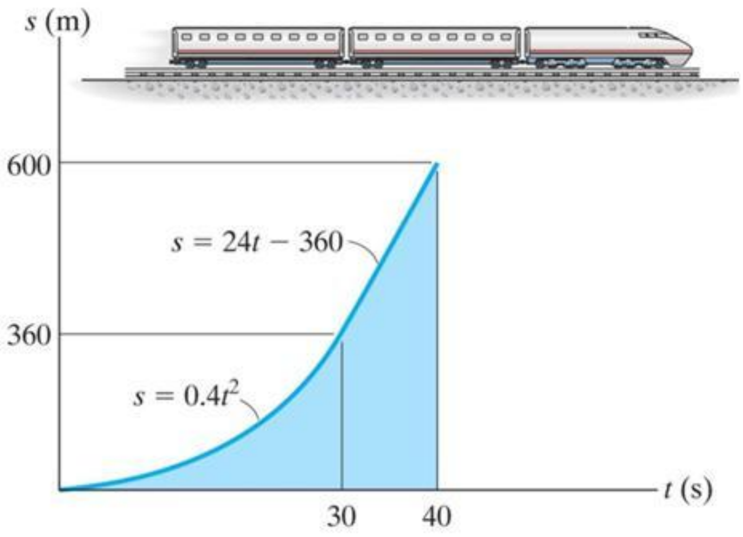
Chapter 12: Kinematics of a Particle

Section 12.3: Rectilinear Kinematics: Erratic Motion

Learning Objective

Be able to calculate position, velocity, and acceleration of a particle using graphs.

Erratic Motion

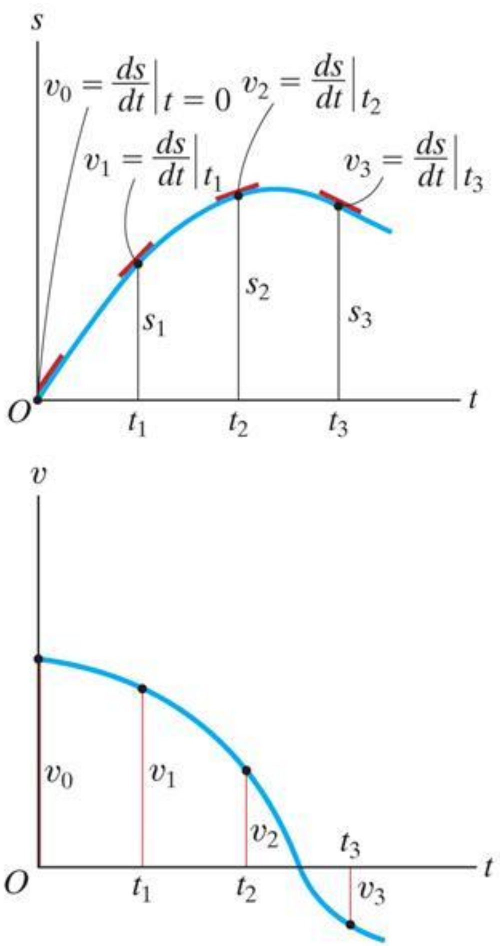


Graphing provides a good way to handle complex motions that would be difficult to describe with formulas.

Graphs also provide a visual description of motion and reinforce the calculus concepts of differentiation and integration as used in dynamics.

The approach builds on the facts that slope and differentiation are linked and that integration can be thought of as finding the area under a curve.

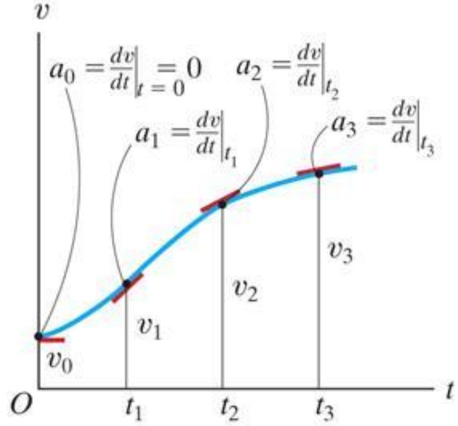
s-t-graph

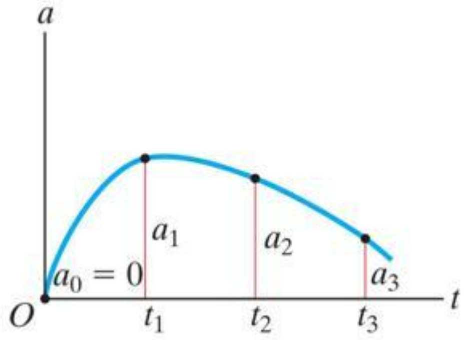


Plots of position vs. time can be used to find velocity vs. time curves. Finding the slope of the line tangent to the motion curve at any point is the velocity at that point (or v = ds/dt).

Therefore, the v-t graph can be constructed by finding the slope at various points along the s-t graph.

v-t-graph



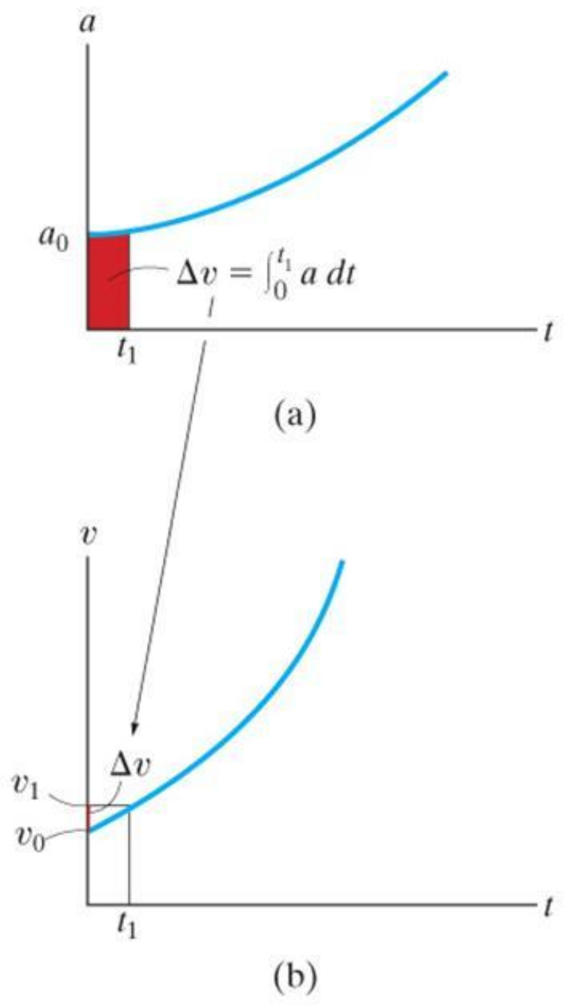


Plots of velocity vs. time can be used to find acceleration vs. time curves. Finding the slope of the line tangent to the velocity curve at any point is the acceleration at that point (or a = dv/dt).

Therefore, the acceleration vs. time (or a-t) graph can be constructed by finding the slope at various points along the v-t graph.

Also, the distance moved (displacement) of the particle is the area under the v-t graph during time Δt.

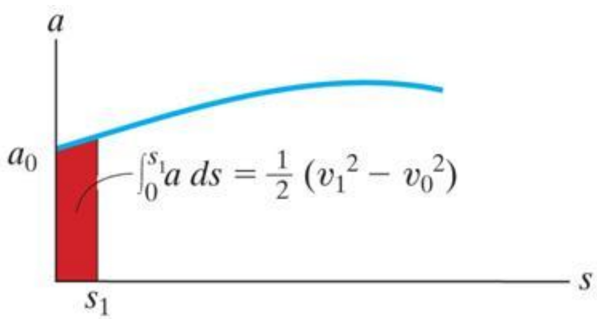
a-t-graph



Given the acceleration vs. time or a-t curve, the change in velocity (Δv) during a time period is the area under the a-t curve.

So we can construct a v-t graph from an a-t graph if we know the initial velocity of the particle.

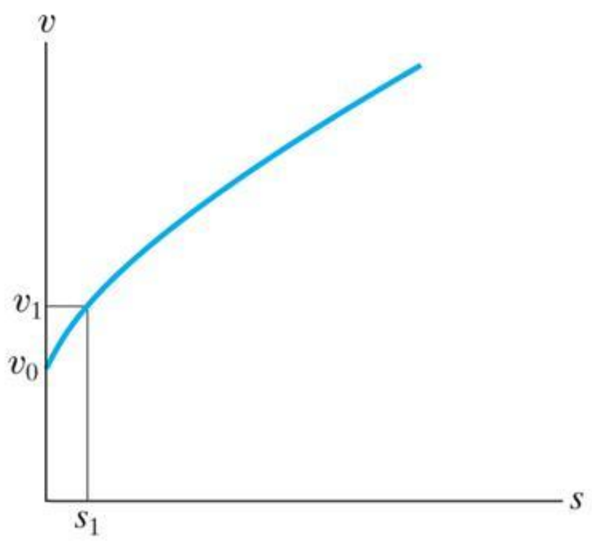
a-s-graph



A more complex case is presented by the acceleration versus position or a-s graph. The area under the a-s curve represents the change in velocity

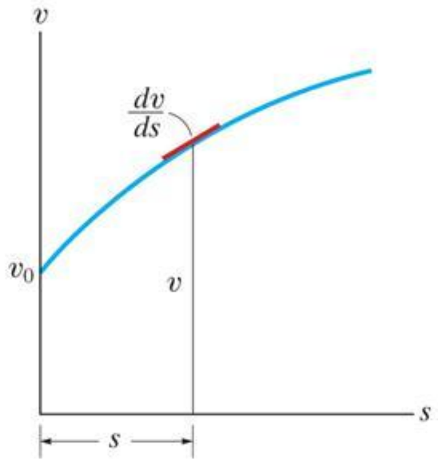
(recall $\int a\,ds = \int v\,dv$).

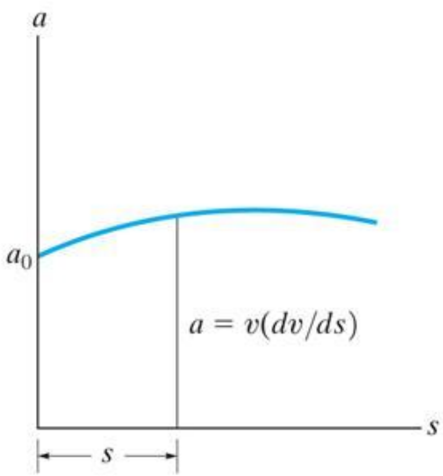
$\frac{1}{2}(v_1^2 - v_o^2) = \int_{s_1}^{s_2} a\,ds =$ area under the a-s graph



This equation can be solved for v_1, allowing you to solve for the velocity at a point. By doing this repeatedly, you can create a plot of velocity versus distance.

v-s-graph





Another complex case is presented by the velocity vs. distance or v-s graph. By reading the velocity v at a point on the curve and multiplying it by the slope of the curve (dv/ds) at this same point, we can obtain the acceleration at that point. Recall the formula

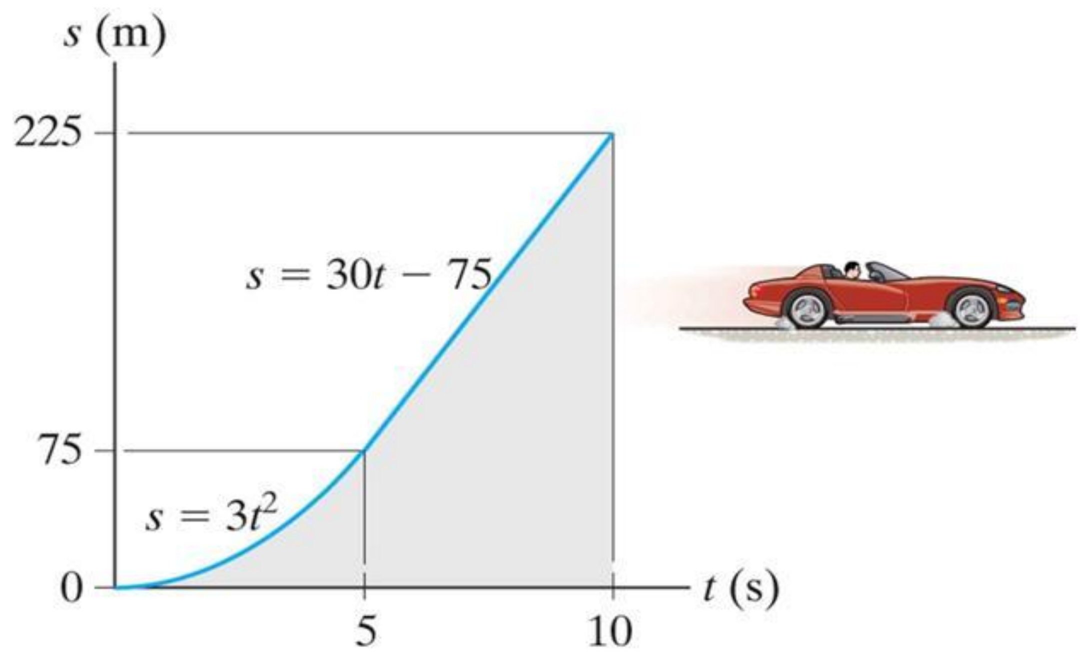
$a = v\,(dv/ds)$.

Thus, we can obtain an a-s plot from the v-s curve.

Example

Given: The s-t graph for a sports car moving along a straight road.

Find: The v-t graph and a-t graph over the time interval shown.



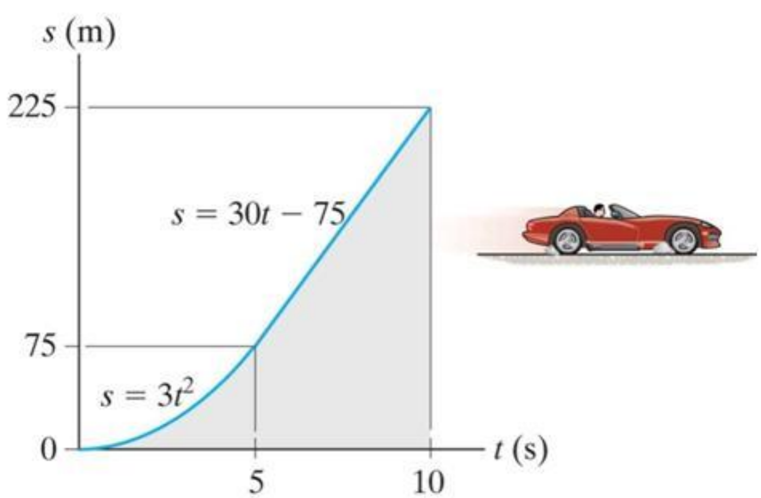
What is your plan of attack for the problem?

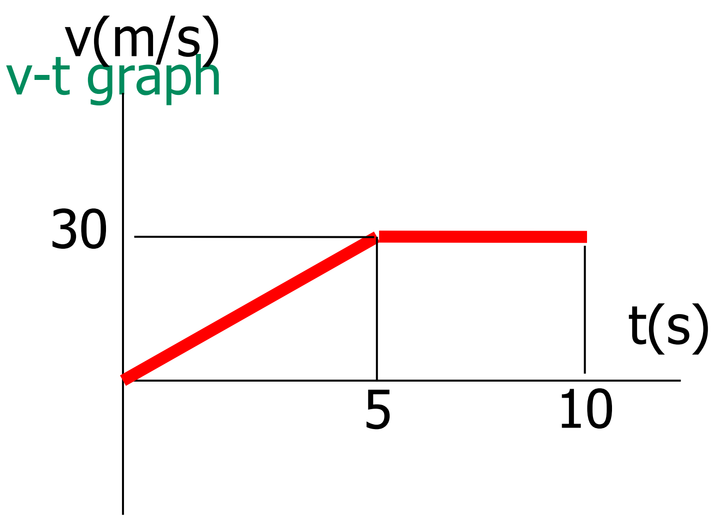
Solution

The v-t graph can be constructed by finding the slope of the s-t graph at key points. What are those?

when $0 < t < 5$ s; $v_{0\text{-}5} = ds/dt = d(3t^2)/dt = 6t$ m/s

when $5 < t < 10$ s; $v_{5\text{-}10} = ds/dt = d(30t-75)/dt = 30$ m/s



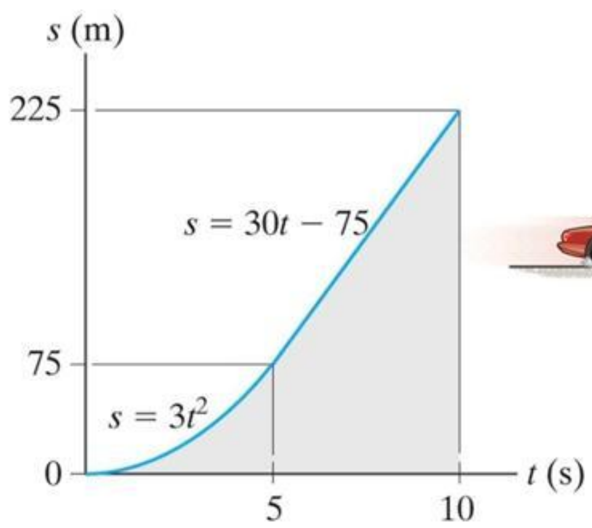


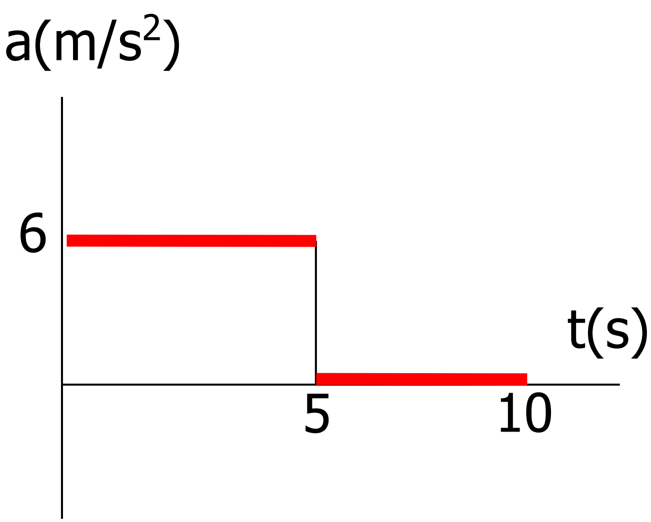
Solution

Similarly, the a-t graph can be constructed by finding the slope at various points along the v-t graph. Using the results of the first part where the velocity was found:

when $0 < t < 5$ s; $a_{0\text{-}5} = dv/dt = d(6t)/dt = 6 \text{ m/s}^2$

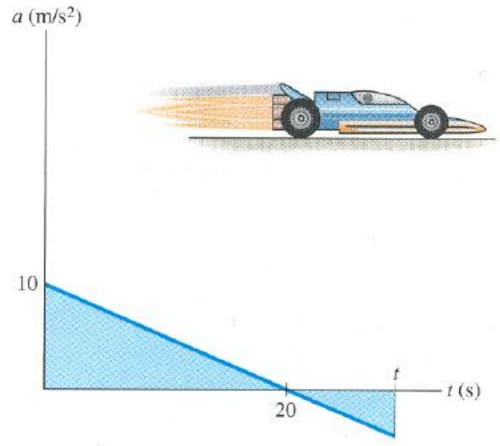
when $5 < t < 10$ s; $a_{5\text{-}10} = dv/dt = d(30)/dt = 0 \text{ m/s}^2$





Quiz

If a particle starts from rest and accelerates according to the graph shown,
the particle's velocity at t = 20 s is

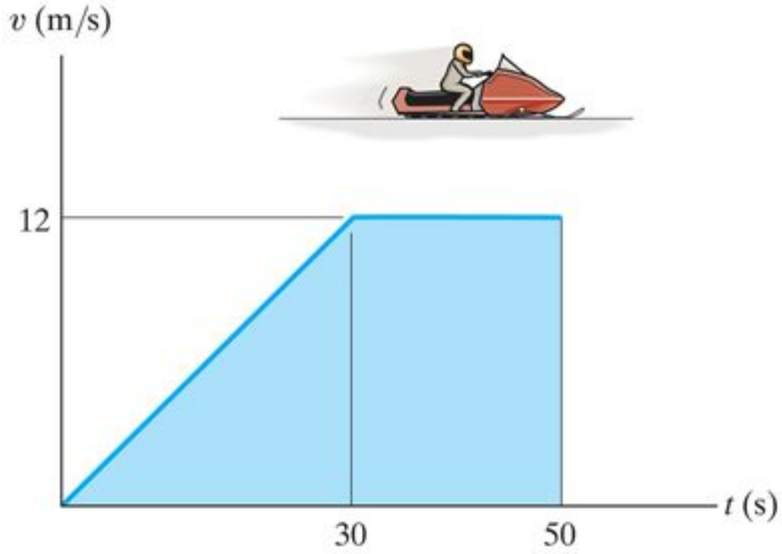


1. 200 m/s
2. 100 m/s
3. 0
4. 20 m/s

The particle in the previous stops moving at t = …….

1. 10 s
2. 20 s
3. 30 s
4. 40 s

Example

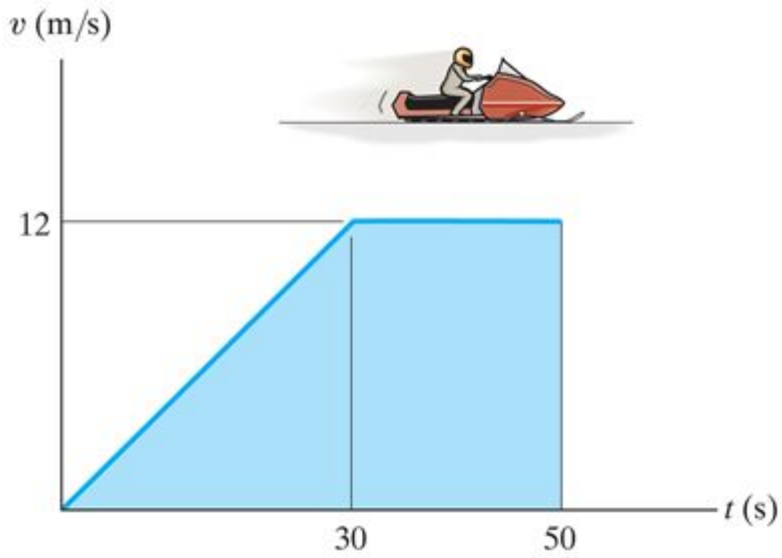


Given: The v-t graph shown.

Find: The a-t graph, average speed, and distance traveled for the 0 - 50 s interval.

Plan: What is your plan?

Example



Given: The v-t graph shown.

Find: The a-t graph, average speed, and distance traveled for the 0 - 50 s interval.

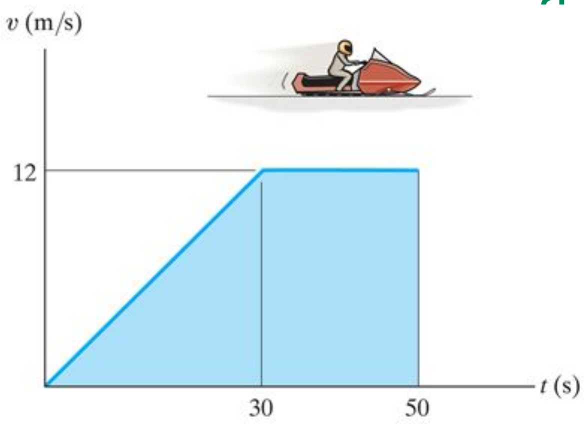
Plan: Find slopes of the v-t curve and draw the a-t graph.
Find the area under the curve. It is the distance traveled.
Finally, calculate average speed (using basic definitions!).

Solution

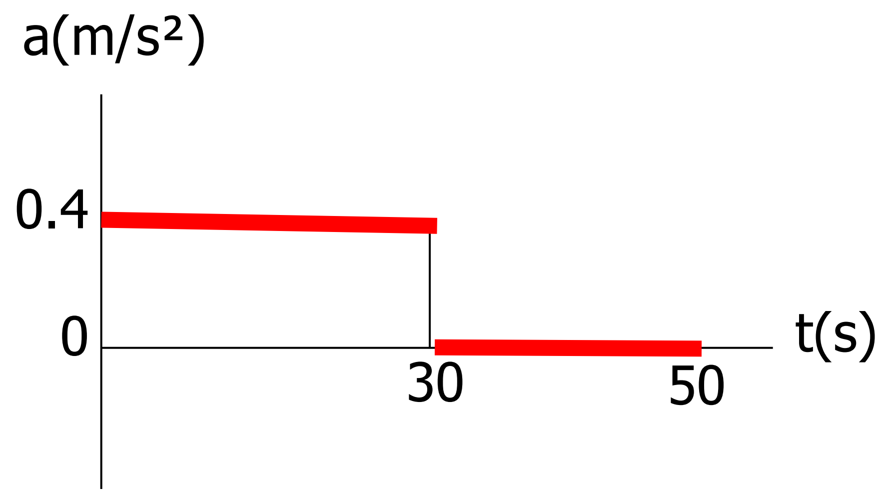
Find the a–t graph:

For $0 \le t \le 30$ $a = dv/dt = 0.4 \ m/s^2$

For $30 \le t \le 50$ $a = dv/dt = 0 \ m/s^2$



a-t graph



Solution

Now find the distance traveled:

$$\Delta s_{0\text{-}30} = \int v\,dt = \int 0.4\,t\,dt = 0.4\,(1/2)\,(30)^2 = 180\ \text{m}$$

$$\Delta s_{30\text{-}50} = \int v\,dt$$

$$= \int 12\,dt = 12\,(50 - 30)$$

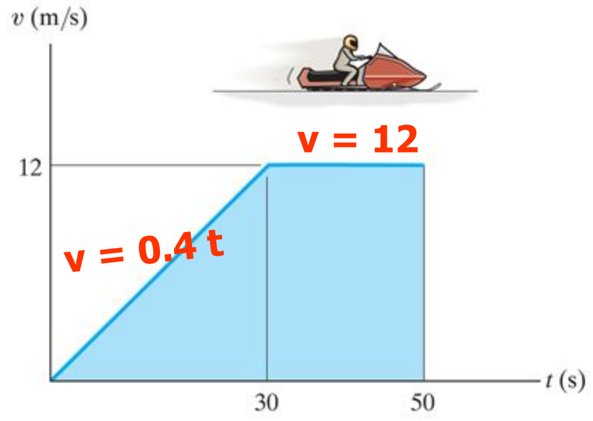
$$= 240\ \text{m}$$

$$s_{0\text{-}50} = 180 + 240 = 420\ \text{m}$$

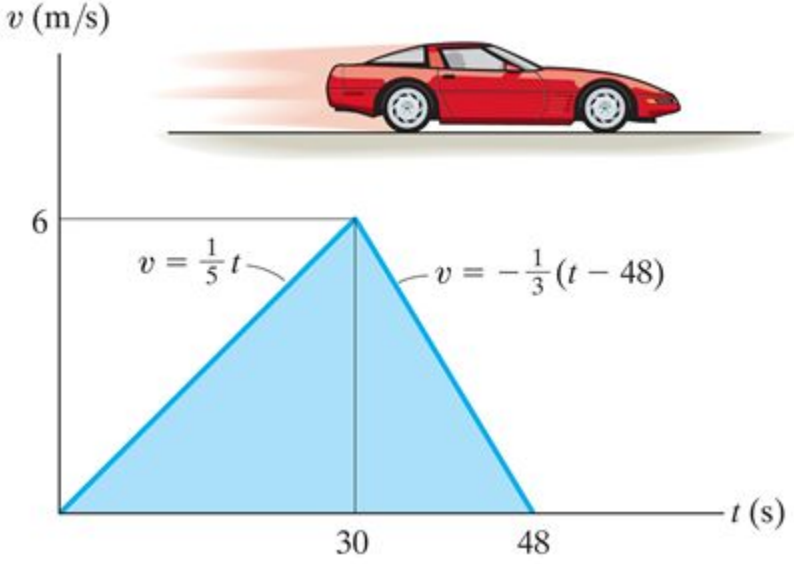
$$v_{avg(0\text{-}50)} = \text{total distance / time}$$

$$= 420 / 50$$

$$= 8.4\ \text{m/s}$$



Example



Given: The v-t graph shown.

Find: The a-t graph, average speed, and distance traveled for the 0 - 48 s interval.

Plan:

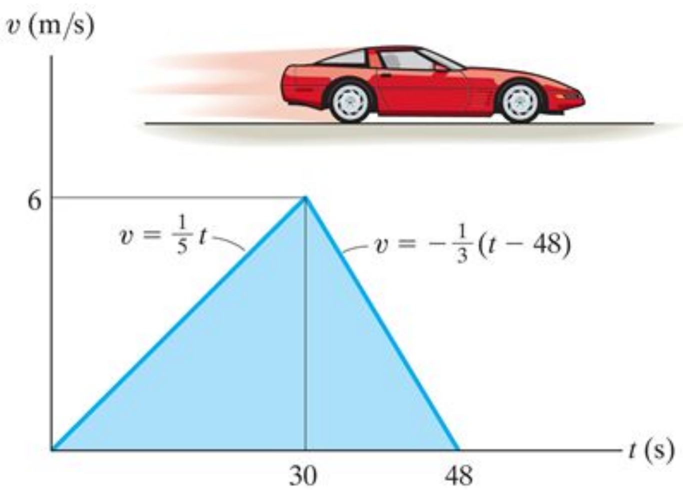
Plan: Find slopes of the v-t curve and draw the a-t graph.
Find the area under the curve. It is the distance traveled.
Finally, calculate average speed (using basic definitions!).

Solution

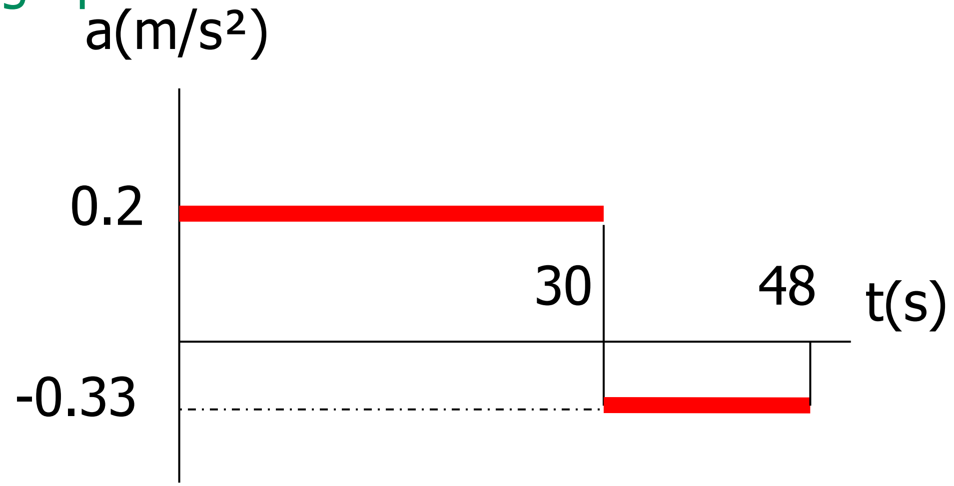
Find the a–t graph:

For $0 \le t \le 30$ $\quad a = dv/dt = 0.2\ m/s^2$

For $30 \le t \le 48$ $\quad a = dv/dt = -0.333\ m/s^2$



a-t graph



Solution

Now find the distance traveled:

$$\Delta s_{0-30} = \int v\,dt = (1/5)(1/2)\,(30)^2 = 90 \text{ m}$$

$$\Delta s_{30-48} = \int v\,dt = \left[(-1/3)\,(1/2)\,(t - 48)^2\right]_{30}^{48}$$

$$= (-1/3)\,(1/2)(48 - 48)^2 - (-1/3)\,(1/2)(30 - 48)^2$$

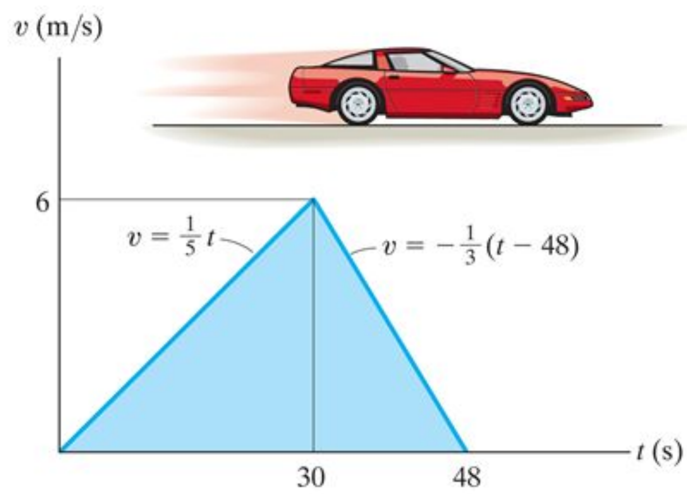
$$= 54 \text{ m}$$

$$s_{0-48} = 90 + 54 = 144 \text{ m}$$

$$v_{avg(0-48)} = \text{total distance / time}$$

$$= 144 \text{ m}/\ 48 \text{ s}$$

$$= 3 \text{ m/s}$$



Example

Given: An aircraft is accelerating whilst taxiing. It starts with a speed of 2 m/s. The acceleration is given as $a = 30\ v^{-4}$ [m/s^2]

Find: Determine the velocity and the distance covered after 40 s

Plan:

Plan: For the determining of the velocity acknowledge the fact that the acceleration has been given as a function of velocity
=> use $a = dv/dt$
For determining the distance acknowledge that we are now looking for distance s while a has been given
=> use $a\ ds = v\ dv$

Solution

a = dv/dt <=> dt = dv/a

$$\int_0^{40} dt = \int_2^{v} \frac{v^4 dv}{30}$$

$40 = 1/150(v^5-32)$

v = 5.7 m/s

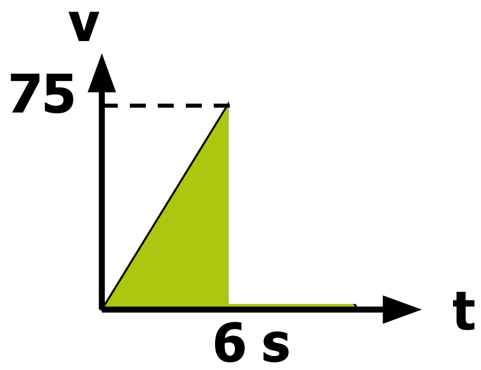
a ds = v dv <=> ds = vdv/a = 1/30 $*v^5$ dv

$$\int_0^{s} ds = \int_2^{5.7} \frac{v^5 dv}{30}$$

$$s = \frac{1}{180}(5.7^6 - 2^6) = 190.8 \text{ m}$$

Quiz

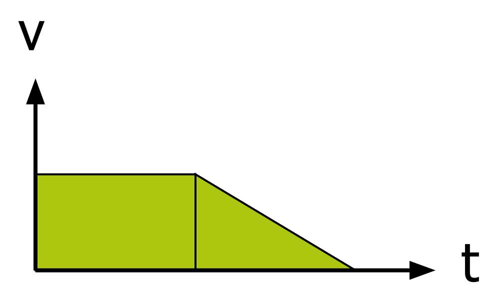
If a car has the velocity curve shown, determine the time t necessary for the car to travel 100 meters.

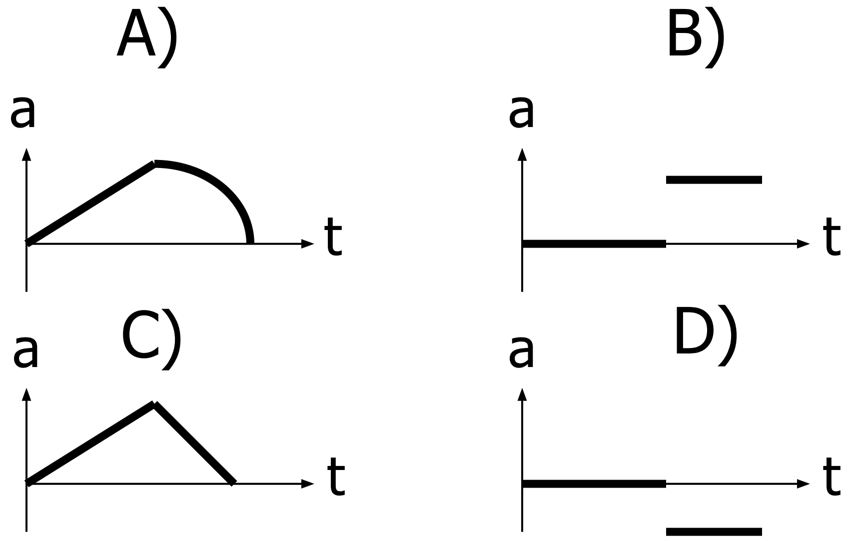


1. 8 s
2. 4 s
3. 10 s
4. 6 s

Select the correct a-t graph for the velocity curve shown.

1. A)
2. B)
3. C)
4. D)





Ugly aircraft competition

Miles M.35 Libellula (1942)

1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10

Chapter 12:
Kinematics of a Particle

Section 12.4: General Curvilinear Motion

Learning Objective

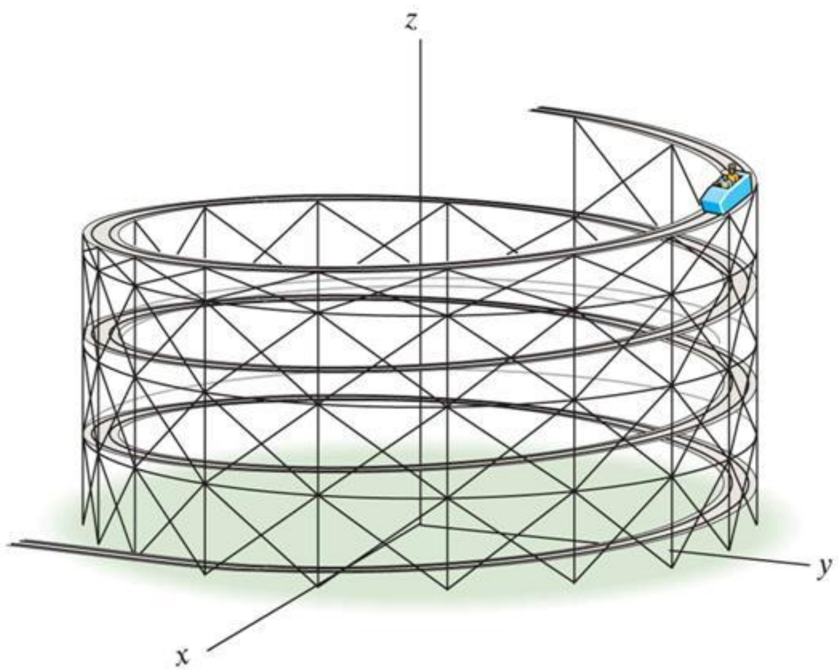
Be able to describe the motion of a particle traveling along a curved path.

Applications

The path of motion of a plane can be tracked with radar and its x, y, and z coordinates (relative to a point on earth) recorded as a function of time.

How can we determine the velocity or acceleration of the plane at any instant?

Applications



A roller coaster car travels down a fixed, helical path at a constant speed.

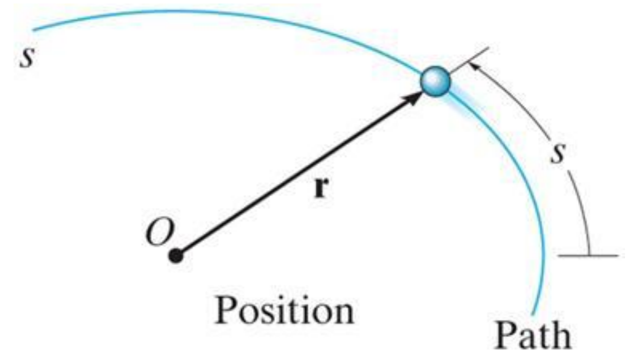
How can we determine its position or acceleration at any instant?

If you are designing the track, why is it important to be able to predict the acceleration of the car?

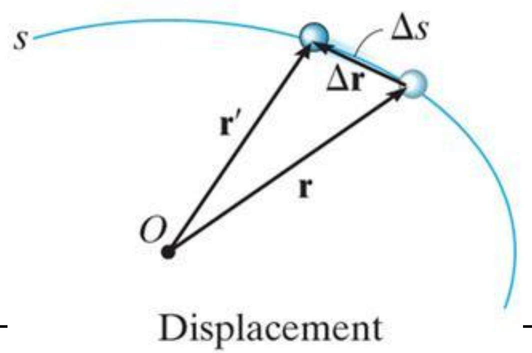
General Curvilinear Motion

A particle moving along a curved path undergoes curvilinear motion. Since the motion is often three-dimensional, **vectors** are used to describe the motion.

A particle moves along a curve defined by the path function, s.



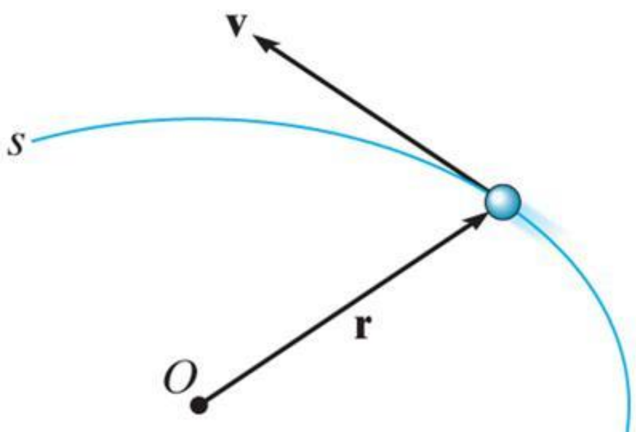
The position of the particle at any instant is designated by the vector ***r*** = ***r***(t). Both the magnitude and direction of ***r*** may vary with time.



If the particle moves a distance Δs along the curve during time interval Δt, the displacement is determined by vector subtraction: $\Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}$

Velocity

Velocity represents the rate of change in the position of a particle.



Velocity

The average velocity of the particle during the time increment Δt is $\boldsymbol{v}_{avg} = \Delta\boldsymbol{r}/\Delta t$.

The instantaneous velocity is the time-derivative of position $\boldsymbol{v} = d\boldsymbol{r}/dt$.

The velocity vector, $\boldsymbol{v}$, is **always** tangent to the path of motion.

The magnitude of $\boldsymbol{v}$ is called the speed. Since the arc length Δs approaches the magnitude of $\Delta\boldsymbol{r}$ as $t \rightarrow 0$, the speed can be obtained by differentiating the path function ($v = ds/dt$). Note that this is not a vector!

Acceleration

Acceleration represents the rate of change in the velocity of a particle.

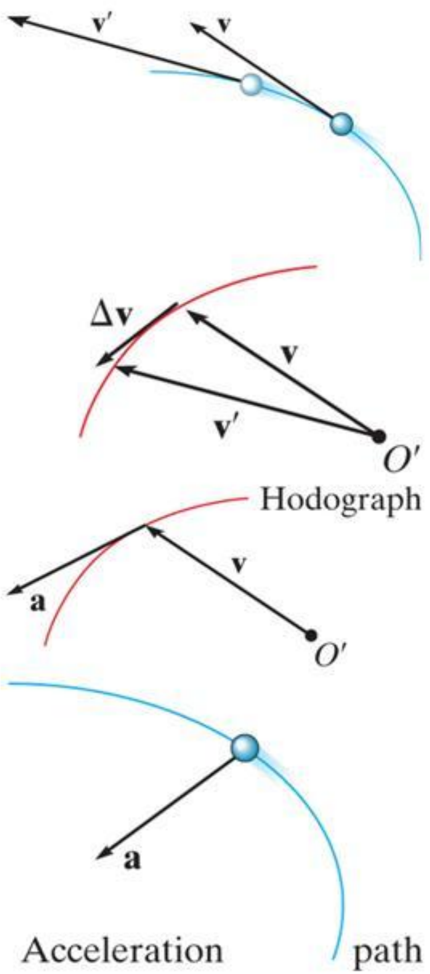
If a particle's velocity changes from $\boldsymbol{v}$ to $\boldsymbol{v}'$ over a time increment Δt, the average acceleration during that increment is:

$$\boldsymbol{a}_{avg} = \Delta\boldsymbol{v}/\Delta t = (\boldsymbol{v}' - \boldsymbol{v})/\Delta t$$

The instantaneous acceleration is the time-derivative of velocity:

$$\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$$

A plot of the locus of points defined by the arrowhead of the velocity vector is called a hodograph.
The acceleration vector is tangent to the hodograph, but not, in general, tangent to the path function.



Chapter 12:
Kinematics of a Particle

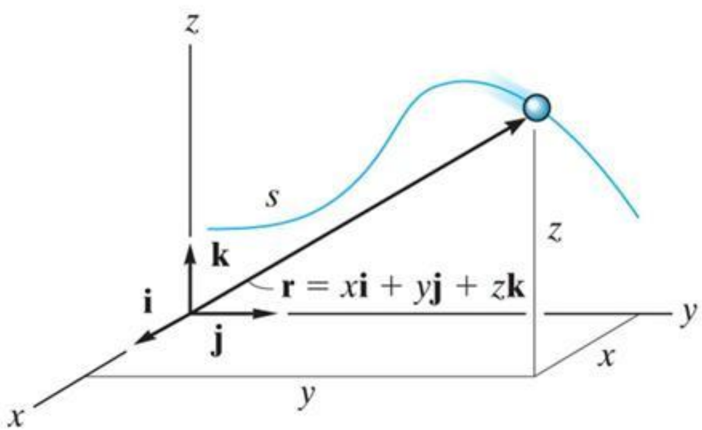
Section 12.5: Curvilinear Motion
Rectangular Components

Learning Objective

Be able to relate kinematic quantities in terms of the rectangular components of the vectors.

Rectangular Components

It is often convenient to describe the motion of a particle in terms of its x, y, z or rectangular components, relative to a fixed frame of reference.



Position

The position of the particle can be defined at any instant by the position vector

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\ .$$

The x, y, z components may all be functions of time, i.e., $x = x(t)$, $y = y(t)$, and $z = z(t)$.

The magnitude of the position vector is: $r = \sqrt{x^2+y^2+z^2}$

The direction of $\boldsymbol{r}$ is defined by the unit vector: $\boldsymbol{u}_r = (1/r)\boldsymbol{r}$

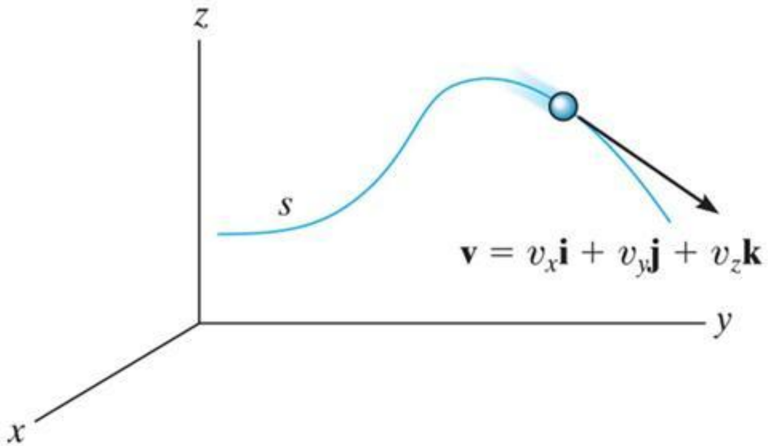
Rectangular Components: Velocity

The velocity vector is the time derivative of the position vector:

$$\boldsymbol{v} = d\boldsymbol{r}/dt = d(x\boldsymbol{i})/dt + d(y\boldsymbol{j})/dt + d(z\boldsymbol{k})/dt$$

Since the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are constant in magnitude and direction, this equation reduces to $v = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}$

where $v_x = \dot{x} = dx/dt$, $v_y = \dot{y} = dy/dt$, $v_z = \dot{z} = dz/dt$



Velocity

The magnitude of the velocity vector is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The direction of $\boldsymbol{v}$ is tangent to the path of motion.

Rectangular Components: Acceleration

The acceleration vector is the time derivative of the velocity vector (second derivative of the position vector):

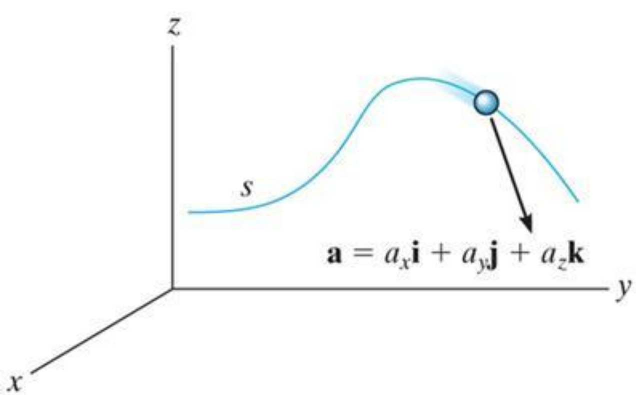
$$\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2 = a_x\,\boldsymbol{i} + a_y\,\boldsymbol{j} + a_z\,\boldsymbol{k}$$

Where
$$a_x = \dot{v}_x = dv_x/dt = \ddot{x}$$
$$a_y = \dot{v}_y = dv_y/dt = \ddot{y}$$
$$a_z = \dot{v}_z = dv_z/dt = \ddot{z}$$

The magnitude of the acceleration vector is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

The direction of $\boldsymbol{a}$ is usually not tangent to the path of the particle.



Acceleration

Example

Given: The box slides down the slope described by the equation
$y = (0.05\,x^2)$ m, where x is in meters.
$v_x = -3$ m/s, $a_x = -1.5$ m/s^2 at $x = 5$ m.

Find: The y components of the velocity and the acceleration of the box at $x = 5$ m.

Plan: Note that the particle's velocity can be related by taking the first time derivative of the path's equation. And the acceleration can be related by taking the second time derivative of the path's equation.

Take a derivative of the position to find the component of the velocity and the acceleration.

Solution

Find the y-component of the velocity by taking a time derivative of the position $y = 0.05\, x^2$

$=> \dot{y} = 2\,(0.05)\, x\, \dot{x} = 0.1\, x\, \dot{x}$

Find the acceleration component by taking a time derivative of the velocity y

$=> \ddot{y} = 0.1\, \dot{x}\, \dot{x} + 0.1\, x\, \ddot{x}$

Substituting the x-component of the acceleration and velocity at x = 5 into $\dot{y}$ and $\ddot{y}$

Solution

Since $\dot{x} = v_x = -3$ m/s, $\ddot{x} = a_x = -1.5$ m/s^2, at $x = 5$ m

=> $\dot{y} = 2 * 0.05\, x\, \dot{x} = 0.1 * 5 * (-3) = -1.5$ m/s

$\ddot{y} = 0.1\, \dot{x}\, \dot{x} + 0.1\, x\, \ddot{x}$

$= 0.1 * (-3)2 + 0.1*5*(-1.5)$
$= 0.9 – 0.75$
$= 0.15$ m/s^2

At $x = 5$ m

$v_y = -1.5$ m/s $= 1.5$ m/s ↓

$a_y = 0.15$ m/s^2 ↑

Quiz

If the position of a particle is defined by

$\mathbf{r} = [(1.5t^2 + 1)\, \boldsymbol{i} + (4t - 1)\, \boldsymbol{j}]$ (m),

its speed at t = 1 s is

1. 2 m/s
2. 3 m/s
3. 5 m/s
4. 7 m/s

The position of a particle is given as $\boldsymbol{r} = (4t^2\,\boldsymbol{i} - 2x\,\boldsymbol{j})$ m.
Determine the particle's acceleration.

1. $(4\,\boldsymbol{i} + 8\,\boldsymbol{j})$ m/s^2
2. $(8\,\boldsymbol{i} - 16\,\boldsymbol{j})$ m/s^2
3. $(8\,\boldsymbol{i})$ m/s^2
4. $(8\,\boldsymbol{j})$ m/s^2

Ugly aircraft competition

Kyushu J7W-1 Shinden (1945)

1. 1
2. 2
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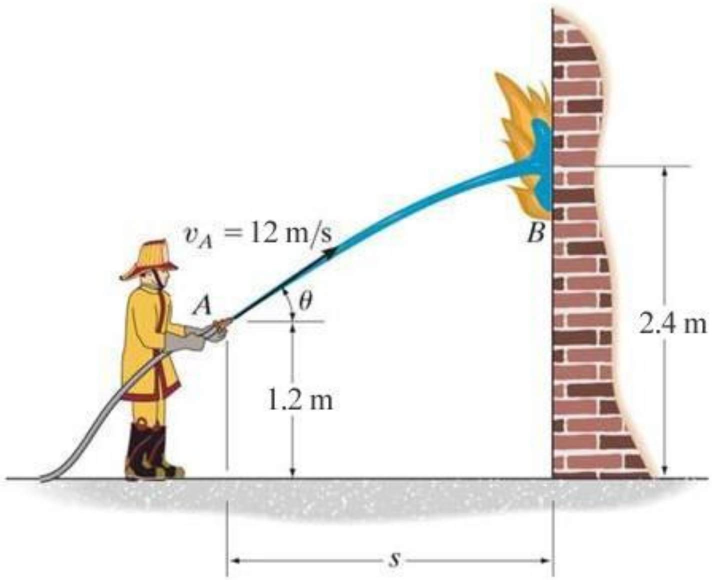
Chapter 12: Kinematics of a Particle

Section 12.6: Motion of a Projectile

Learning Objective

Be able to analyze the free-flight motion of a projectile.

Applications

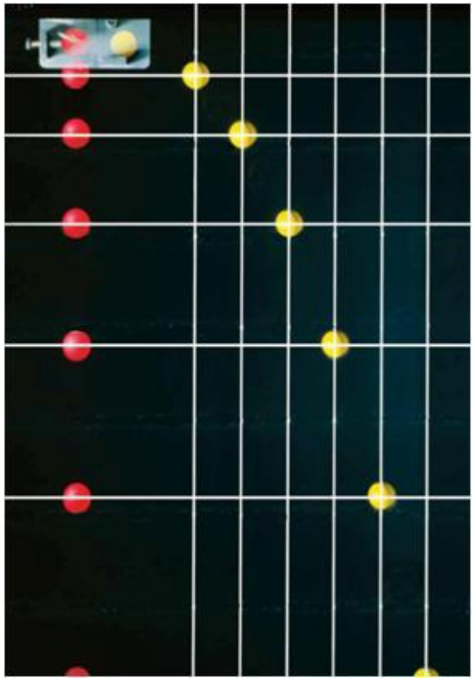


A firefighter needs to know the maximum height on the wall she can project water from the hose. What parameters would you program into a wrist computer to find the angle, θ, that she should use to hold the hose?

Motion of a Projectile

Projectile motion can be treated as two rectilinear motions, one in the horizontal direction experiencing zero acceleration and the other in the vertical direction experiencing constant acceleration (i.e. from gravity).

Motion of a Projectile

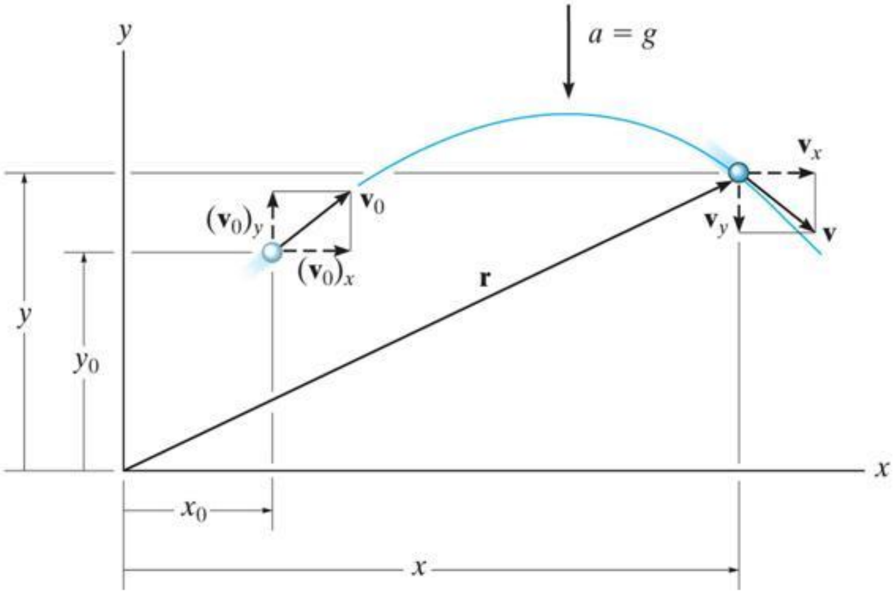


For illustration, consider the two balls on the left. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity. Each picture in this sequence is taken after the same time interval.

Notice both balls are subjected to the same downward acceleration since they remain at the same elevation at any instant.

Also, note that the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction is constant.

Kinematic Equations: Horizontal Motion



Since $a_x = 0$, the velocity in the horizontal direction remains constant ($v_x = v_{ox}$) and the position in the x direction can be determined by:
$x = x_o + (v_{ox})\ t$

Why is a_x equal to zero (what assumption must be made if the movement is through the air)?

Kinematic Equations: Vertical Motion

Since the positive y-axis is directed upward, $a_y = -g$. Application of the constant acceleration equations yields:

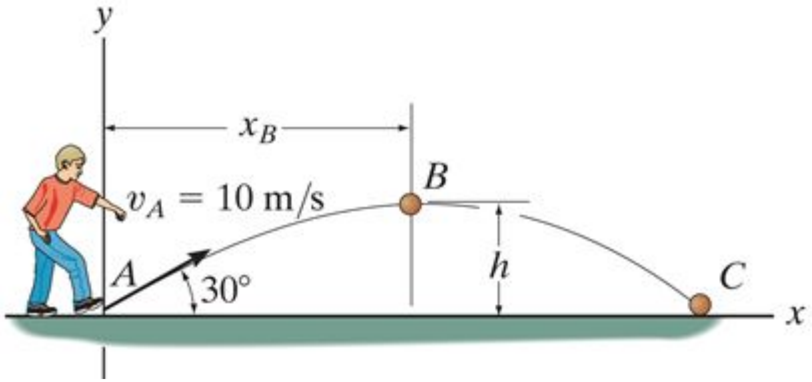
$$v_y = v_{oy} - g\,t$$

$$y = y_o + (v_{oy})\,t - \tfrac{1}{2}\,g\,t^2$$

$$v_y^{\,2} = v_{oy}^{\,2} - 2\,g\,(y - y_o)$$

For any given problem, only two of these three equations can be used. Why?

Example



Given: v_A and θ

Find: Horizontal distance it travels and v_C.

Plan: Apply the kinematic relations in x- and y-directions.

Solution: Using $v_{Ax} = 10 \cos 30$ and $v_{Ay} = 10 \sin 30$

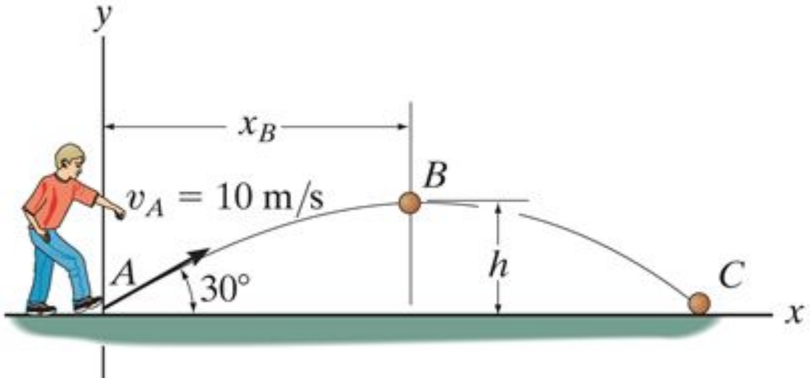
We can write:

$$v_x = 10 \cos 30$$
$$v_y = 10 \sin 30 - (9.81)\, t$$
$$x = (10 \cos 30)\, t$$
$$y = (10 \sin 30)\, t - ½\, (9.81)\, t^2$$

Since y = 0 at C

$$0 = (10 \sin 30)\, t - ½\, (9.81)\, t^2 \Rightarrow t = 0 \text{ and } 1.019 \text{ s}$$

Solution



Velocity components at C are;

$v_{Cx} = 10 \cos 30$
$= 8.66$ m/s $\rightarrow$

$v_{Cy} = 10 \sin 30 - (9.81)(1.019)$
$= -5$ m/s $= 5$ m/s $\downarrow$

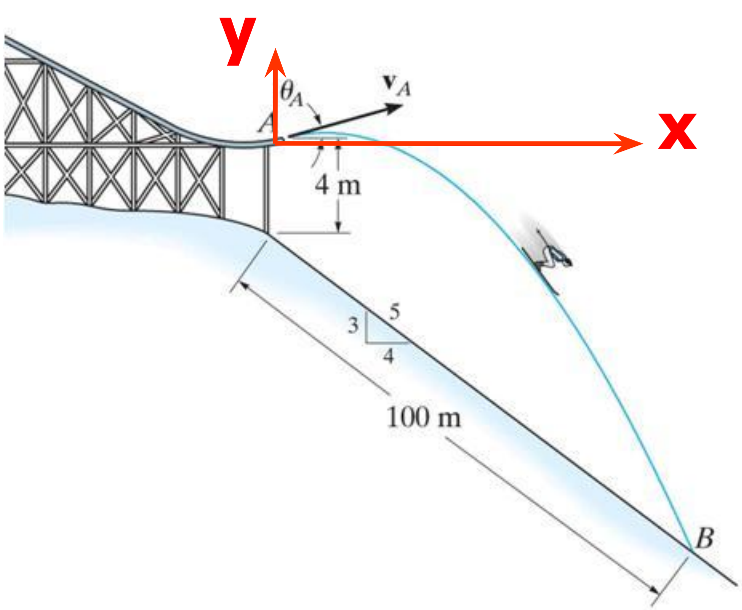
$$v_C = \sqrt{8.66^2 + (-5)^2} = 10 \text{ m/s}$$

Horizontal distance the ball travels is;

$x = (10 \cos 30)\, t$

$x = (10 \cos 30)\, 1.019 = 8.83$ m

Example



Given: A skier leaves the ski jump ramp at $\theta_A = 25^o$ and hits the slope at B.

Find: The skier's initial speed v_A.

Plan:

Plan: Establish a fixed x,y coordinate system (in this solution, the origin of the coordinate system is placed at A). Apply the kinematic relations in x- and y-directions.

Solution

Motion in x-direction:

Using $x_B = x_A + v_{ox}(t_{AB})$ => $(4/5)100 = 0 + v_A (\cos 25°)\, t_{AB}$

$$t_{AB} = \frac{80}{v_A (\cos 25°)} = \frac{88.27}{v_A}$$

Motion in y-direction:

Using $y_B = y_A + v_{oy}(t_{AB}) - ½\, g(t_{AB})^2$

$$-64 = 0 + v_A(\sin 25°)\left\{\frac{88.27}{v_A}\right\} - ½\,(9.81)\left\{\frac{88.27}{v_A}\right\}^2$$

$v_A = 19.42$ m/s

Quiz

The time of flight of a projectile, fired over level ground, with initial velocity V_o at angle θ, is equal to?

1. $(v_o \sin \theta)/g$
2. $(2v_o \sin \theta)/g$
3. $(v_o \cos \theta)/g$
4. $(2v_o \cos \theta)/g$

Ugly aircraft competition

VariViggen (1967)

1. 1
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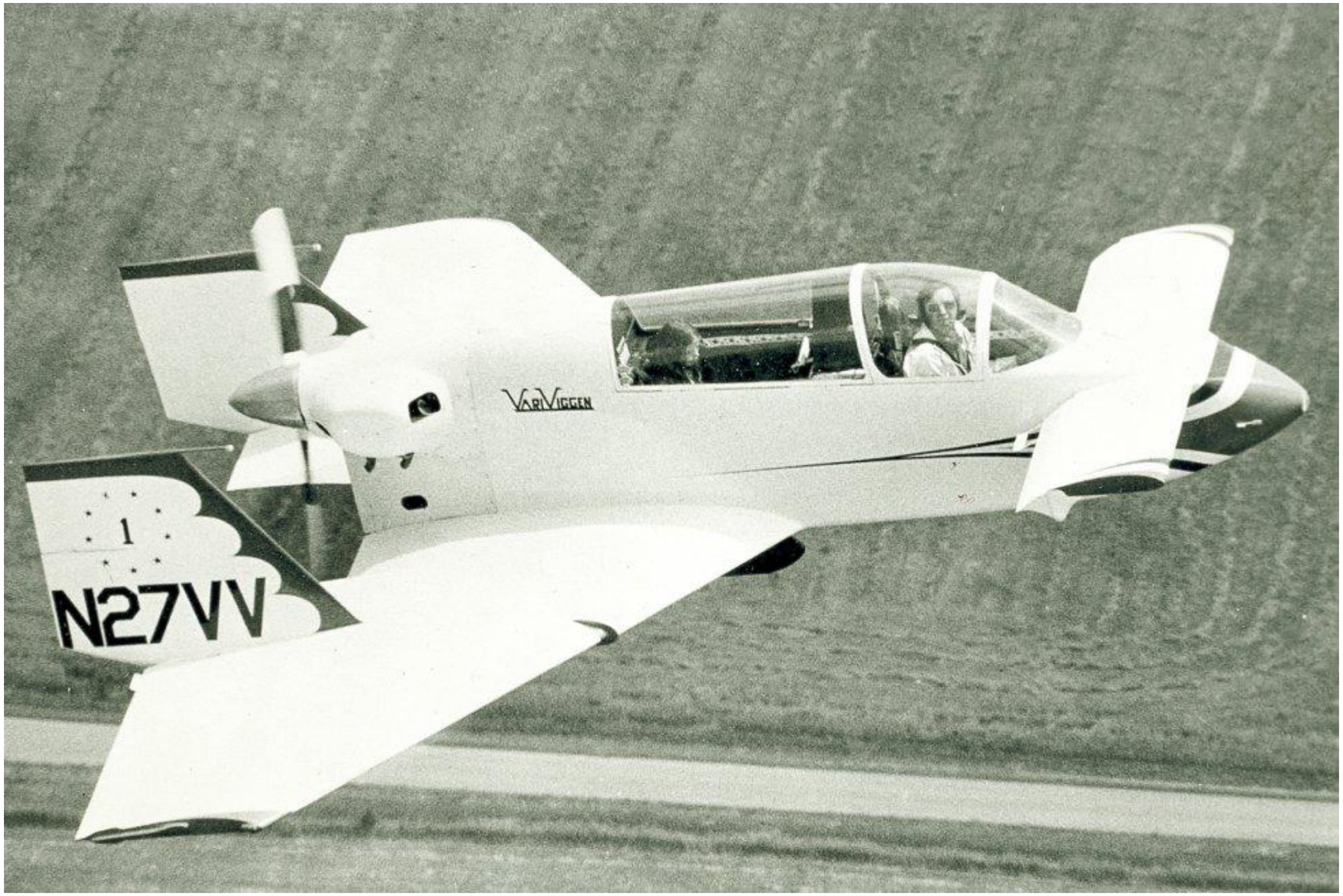


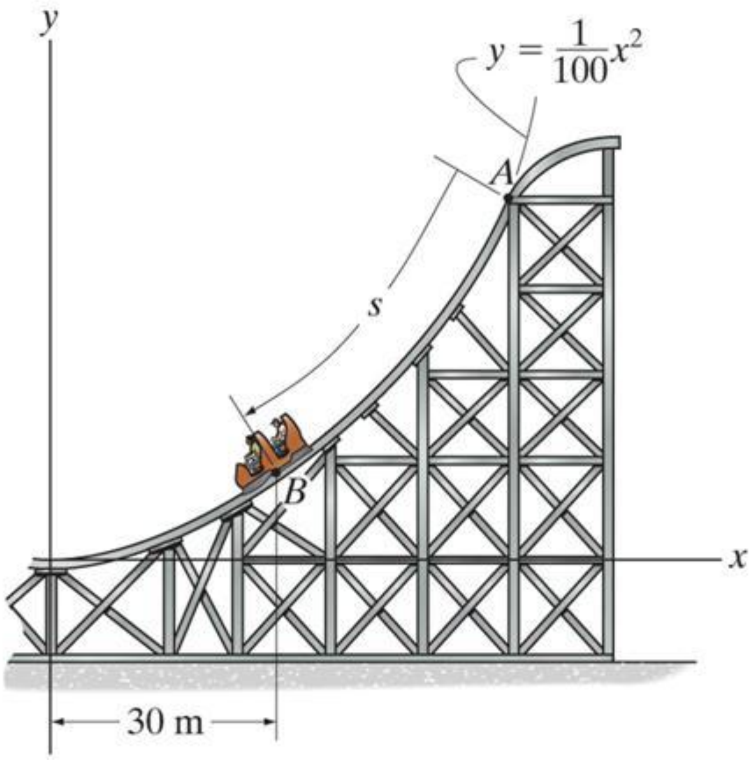
Chapter 12: Kinematics of a Particle

Section 12.7: Curvilinear Motion
Normal and Tangential Components

Learning Objective

Be able to calculate the normal and tangential components of velocity and acceleration of a particle traveling along a curved path.

Application



A roller coaster travels down a hill for which the path can be approximated by a function
y = f(x).

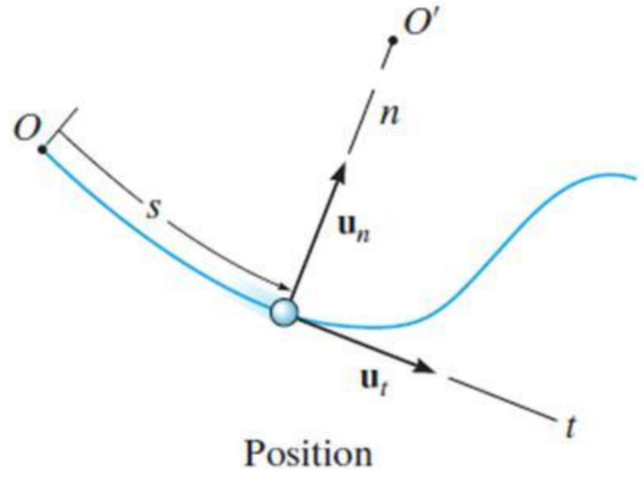
The roller coaster starts from rest and increases its speed at a constant rate.

How can we determine its velocity and acceleration at the bottom?

Normal and Tangential Components

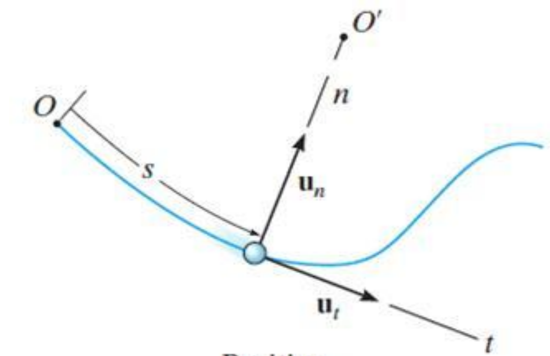
When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal (n) and tangential (t) coordinates are often used.

In the n-t coordinate system, the origin is located on the particle (the origin moves with the particle).



The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion.
The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.

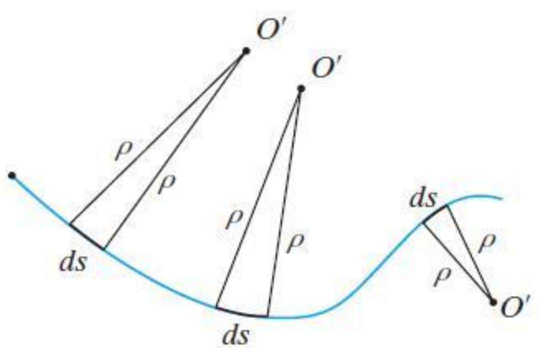
Normal and Tangential Components



Position

The positive n and t directions are defined by the unit vectors $\boldsymbol{u}_n$ and $\boldsymbol{u}_t$, respectively.

The center of curvature, O’, always lies on the concave side of the curve.

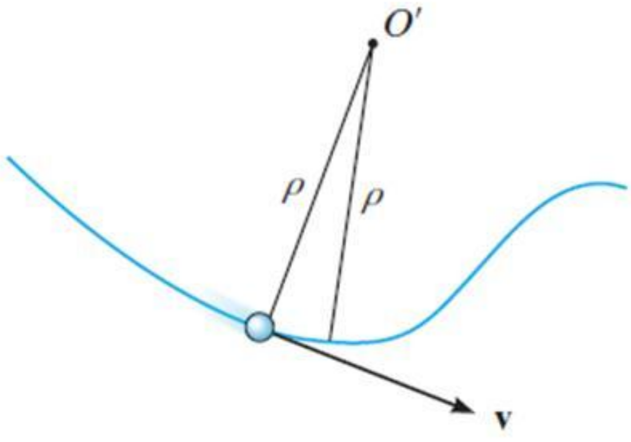


Radius of curvature

The radius of curvature, r, is defined as the perpendicular distance from the curve to the center of curvature at that point.

tion of the particle at any instant is defined by the distance, s, along the curve from a fixed reference point.

Velocity in the n-t-Coordinate System



Velocity

The velocity vector is always tangent to the path of motion (t-direction).

The magnitude is determined by taking the time derivative of the path function, s(t).

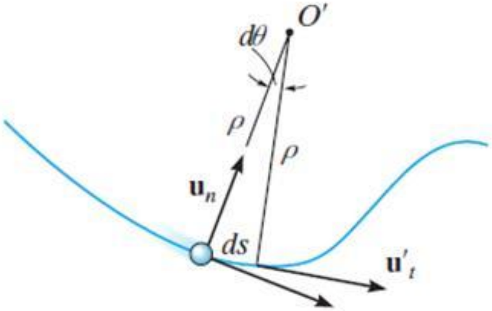
$$\boldsymbol{v} = v\,\boldsymbol{u}_t \quad \text{where} \quad v = \dot{s} = ds/dt$$

Here v defines the magnitude of the velocity (speed) and $\boldsymbol{u}_t$ defines the direction of the velocity vector.

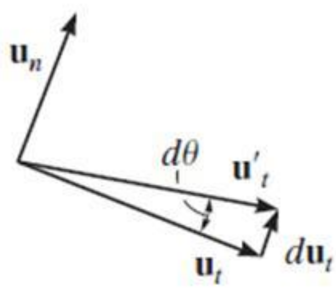
Velocity in the n-t-Coordinate System

Acceleration is the time rate of change of velocity:

$\mathbf{a} = d\boldsymbol{v}/dt = d(v\boldsymbol{u}_t)/dt = \dot{v}\,\boldsymbol{u}_t + v\dot{\boldsymbol{u}}_t$



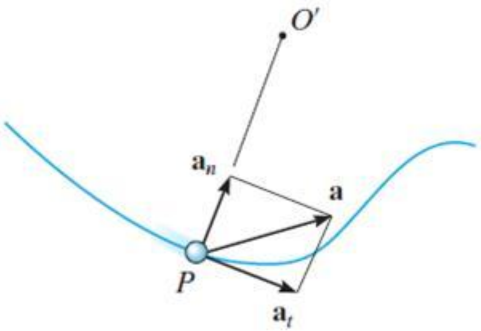
Here $\dot{v}$ represents the change in the magnitude of velocity and $\dot{\boldsymbol{u}}_t$ represents the rate of change in the direction of $\boldsymbol{u}_t$.



After mathematical manipulation, the acceleration vector can be expressed as:

$\boldsymbol{a} = \dot{v}\boldsymbol{u}_t + (v^2/r)\boldsymbol{u}_n = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n$

Velocity in the n-t-Coordinate System



Acceleration

So, there are two components to the acceleration vector:

$$\mathbf{a} = a_t \, \boldsymbol{u}_t + a_n \, \boldsymbol{u}_n$$

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.

  $a_t = \dot{v}$ or $a_t \, ds = v \, dv$

- The normal or centripetal component is always directed toward the center of curvature of the curve. $a_n = v^2/r$

- The magnitude of the acceleration vector is

  $$a = \sqrt{a_t^2 + a_n^2}$$

Special Cases of Motion

There are four special cases of motion to consider.

1) The particle moves along a straight line.

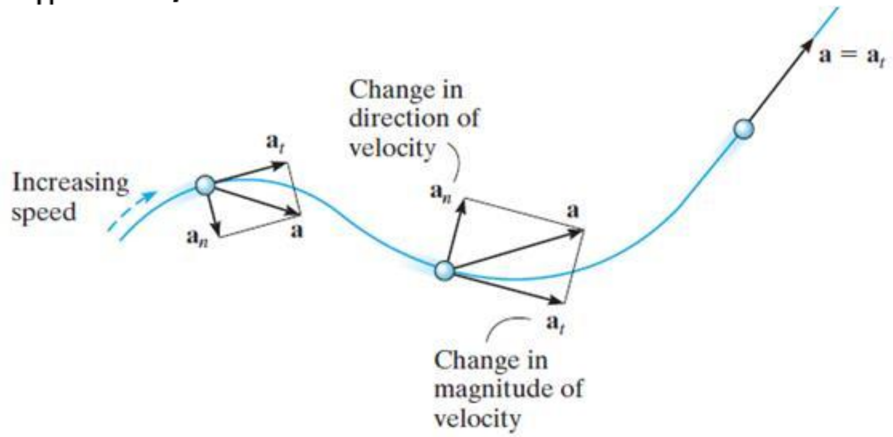
$r \longrightarrow \infty \quad => \quad a_n = v^2/r = 0 \quad => \quad a = a_t = \dot{v}$

The tangential component represents the time rate of change in the magnitude of the velocity.

2) The particle moves along a curve at constant speed.

$a_t = \dot{v} = 0 \quad => \quad a = a_n = v^2/r$

The normal component represents the time rate of change in the direction of the velocity.



Special Cases of Motion

3) The tangential component of acceleration is constant, $a_t = (a_t)_c$.

In this case, $s = s_o + v_o t + (1/2)\,(a_t)_c\, t^2$

$v = v_o + (a_t)_c\, t$

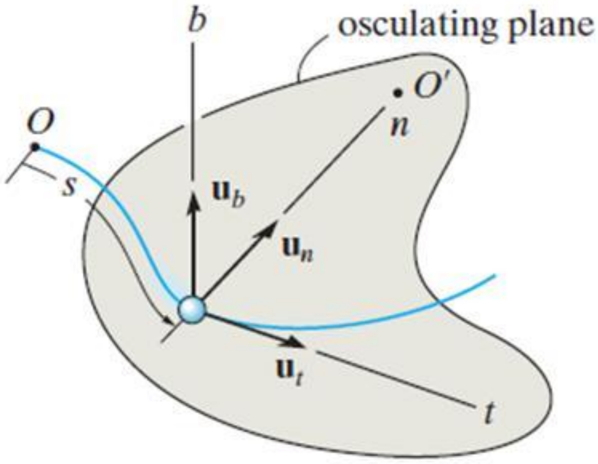
$v^2 = (v_o)^2 + 2\,(a_t)_c\,(s - s_o)$

As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$. How are these equations related to projectile motion equations? Why?

4) The particle moves along a path expressed as $y = f(x)$. The radius of curvature, r, at any point on the path can be calculated from

$$r = \frac{[\,1 + (dy/dx)^2\,]^{3/2}}{\left| d^2y/dx^2 \right|}$$

Three-dimensional Motion

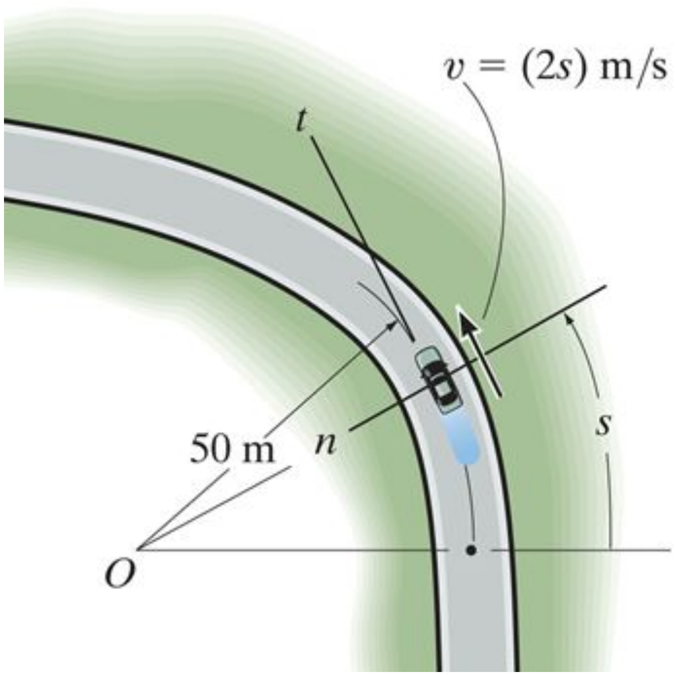


If a particle moves along a space curve, the n and t axes are defined as before. At any point, the t-axis is tangent to the path and the n-axis points toward the center of curvature. The plane containing the n and t axes is called the osculating plane.

A third axis can be defined, called the binomial axis, b. The binomial unit vector, $\boldsymbol{u}_b$, is directed perpendicular to the osculating plane, and its sense is defined by the cross product $\boldsymbol{u}_b = \boldsymbol{u}_t \times \boldsymbol{u}_n$.

In our cases there is no motion, thus no velocity or acceleration, in the binomial direction.

Example



Given: A car travels along the road with a speed of v = (2s) m/s, where s is in meters.
r = 50 m

Find: The magnitudes of the car's acceleration at s = 10 m.

Plan:

1) Calculate the velocity when s = 10 m using v(s).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

Solution

1) The velocity vector is $\boldsymbol{v} = v\, \boldsymbol{u}_t$, where the magnitude is given by $v = (2s)$ m/s. When $s = 10$ m: $v = 20$ m/s

2) The acceleration vector is $\mathbf{a} = a_t \boldsymbol{u}_t + a_n \boldsymbol{u}_n = \dot{v}\, \boldsymbol{u}_t + (v^2/r) \boldsymbol{u}_n$

Tangential component:

Since $a_t = \dot{v} = dv/dt = (dv/ds)\,(ds/dt) = v\,(dv/ds)$

where $v = 2s \quad \Rightarrow \quad a_t = d(2s)/ds\,(v) = 2\,v$

At $s = 10$ m: $a_t = 40$ m/s^2

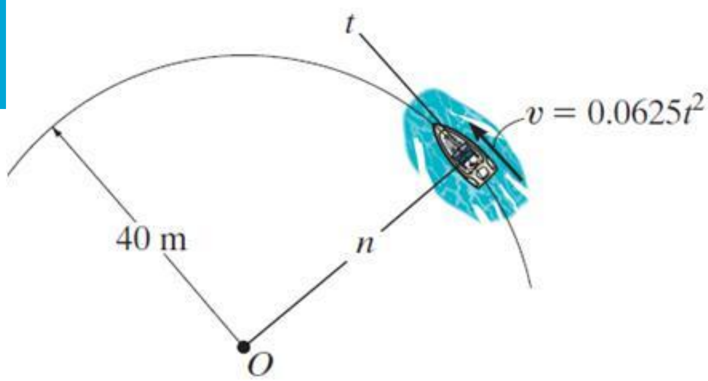
Normal component: $a_n = v^2/r$

When $s = 10$ m: $a_n = (20)^2 / (50) = 8$ m/s^2

The magnitude of the acceleration is

$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(40)^2 + (8)^2]^{0.5} = 40.8$ m/s^2

Example



Given: A boat travels around a circular path, r = 40 m, at a speed that increases with time, $v = (0.0625\ t^2)$ m/s.

Find: The magnitudes of the boat's velocity and acceleration at the instant t = 10 s.

Plan:

The boat starts from rest (v = 0 when t = 0).

1) Calculate the velocity at t = 10 s using v(t).
2) Calculate the tangential and normal components of acceleration and then the magnitude of the acceleration vector.

Solution

1) The velocity vector is $\boldsymbol{v} = v\,\boldsymbol{u}_t$, where the magnitude is given by $v = (0.0625t^2)$ m/s. At t = 10s:

$$v = 0.0625\,t^2 = 0.0625\,(10)^2 = 6.25 \text{ m/s}$$

2) The acceleration vector is $\boldsymbol{a} = a_t\boldsymbol{u}_t + a_n\boldsymbol{u}_n = \dot{v}\boldsymbol{u}_t + (v^2/r)\boldsymbol{u}_n$.

Tangential component: $a_t = \dot{v} = d(0.0625\,t^2)/dt = 0.125\,t$ m/s^2

At t = 10s: $a_t = 0.125t = 0.125(10) =$ 1.25 m/s^2

Normal component: $a_n = v^2/r$ m/s^2

At t = 10 s: $a_n = (6.25)^2 / (40) =$ 0.9766 m/s^2

The magnitude of the acceleration is

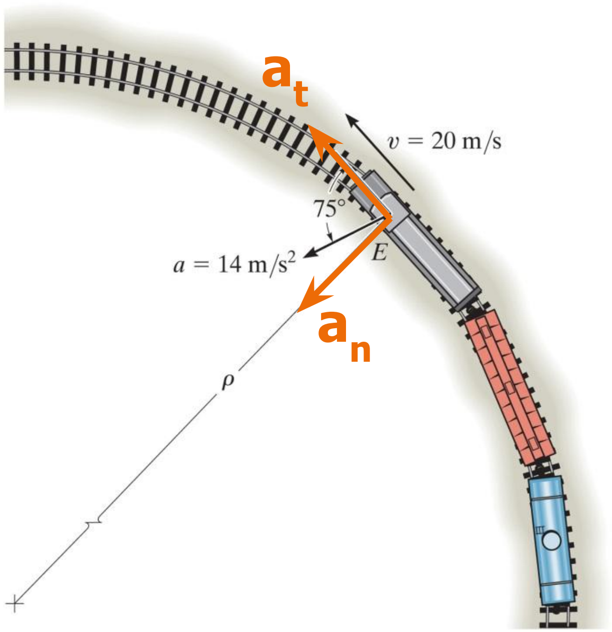
$$a = [(a_t)^2 + (a_n)^2]^{0.5} = [(1.25)^2 + (0.9766)^2]^{0.5} = 1.59 \text{ m/s}^2$$

Quiz

An aircraft traveling in a circular path of radius 300 m has an instantaneous velocity of 30 m/s and its velocity is increasing at a constant rate of 4 m/s^2. What is the magnitude of its total acceleration at this instant?

1. 3 m/s^2
2. 4 m/s^2
3. 5 m/s^2
4. -5 m/s^2

Example



Given: The train engine at E has a speed of 20 m/s and an acceleration of 14 m/s^2 acting in the direction shown.

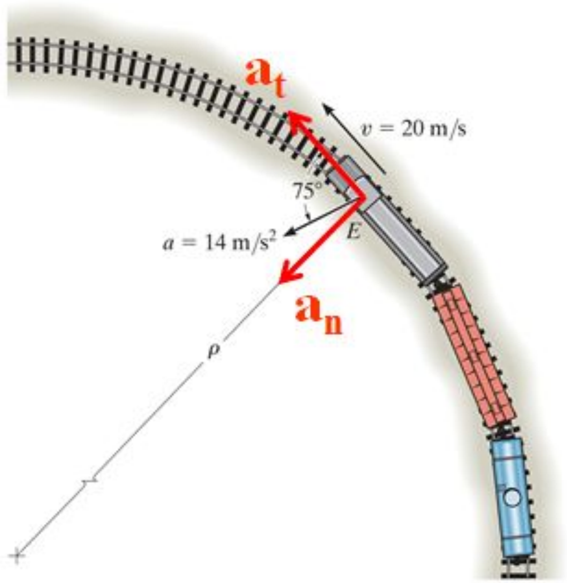
Find: The rate of increase in the train's speed and the radius of curvature ρ of the path.

Plan:

1) Determine the tangential and normal components of the acceleration
2) Calculate dv/dt form the tangential component of the acceleration
3) Calculate ρ from the normal component of the acceleration

Solution

1) Acceleration



Tangential component :

a_t =14 cos(75) = 3.623 m/s^2

Normal component :

a_n = 14 sin(75) = 13.52 m/s^2

2) The tangential component of the acceleration is the rate of increase of the train's speed so

a_t = 3.62 m/s^2

3) The normal component of acceleration is

$a_n = v^2/r \quad \Rightarrow 13.52 = 20^2 / r$

- r = 29.6 m

Chapter 12: Kinematics of a Particle

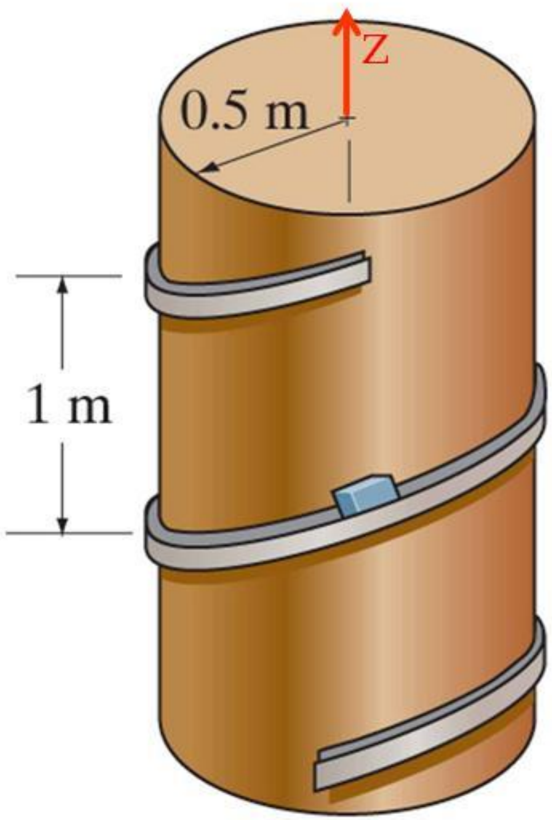
Section 12.8: Curvilinear Motion

Cylindrical Components

Learning Objective

Be able to calculate velocity and acceleration components using cylindrical coordinates.

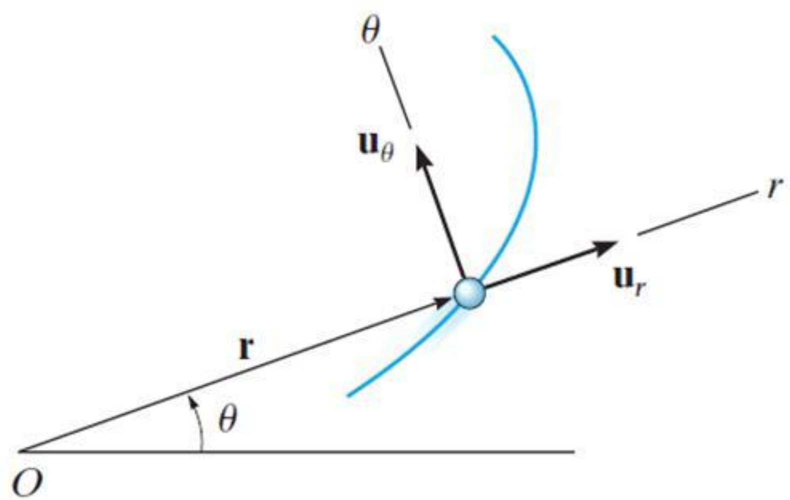
Applications



A cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

In the figure shown, the box slides down the helical ramp. How would you find the box's velocity components to check to see if the package will fly off the ramp?

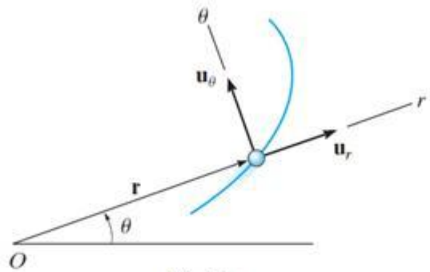
Cylindrical Components



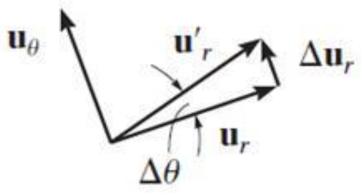
Position

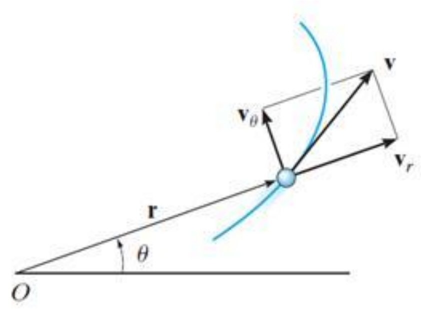
We can express the location of P in polar coordinates as $\mathbf{r} = r\,\mathbf{u}_r$. Note that the radial direction, r, extends outward from the fixed origin, O, and the transverse coordinate, θ, is measured counter-clockwise (CCW) from the horizontal.

Velocity in Polar Coordinates



Position





Velocity

The instantaneous velocity is defined as:

$$\boldsymbol{v} = d\boldsymbol{r}/dt \;=\; d(r\boldsymbol{u_r})/dt$$

$$\boldsymbol{v} = \dot{r}\boldsymbol{u_r} + r\,\frac{d\boldsymbol{u_r}}{dt}$$

Using the chain rule:

$$d\boldsymbol{u_r}/dt = (d\boldsymbol{u_r}/d\theta)(d\theta/dt)$$

We can prove that $d\boldsymbol{u_r}/d\theta = \boldsymbol{u_\theta}$ so $d\boldsymbol{u_r}/dt = \dot{\theta}\boldsymbol{u_\theta}$

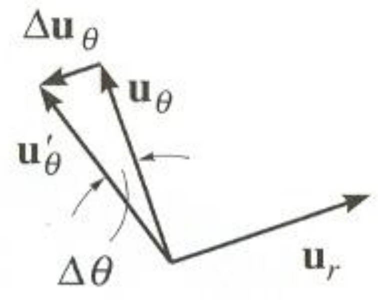
Therefore: $\boldsymbol{v} = \dot{r}\boldsymbol{u_r} + r\dot{\theta}\boldsymbol{u_\theta}$

Thus, the velocity vector has two components: $\dot{r}$, called the radial component, and $r\dot{\theta}$ called the transverse component. The speed of the particle at any given instant is the sum of the squares of both components or

$$v = \sqrt{(r\dot{\theta})^2 + (\dot{r})^2}$$

TU Delft

Acceleration in Polar Coordinates

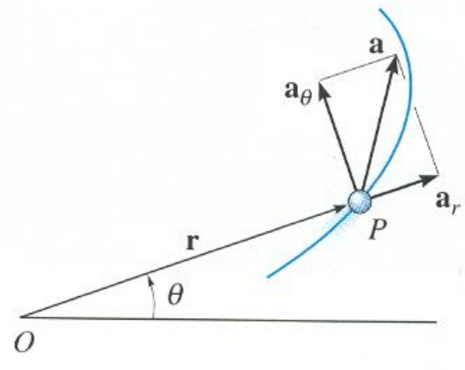


The instantaneous acceleration is defined as:

$$\mathbf{a} = d\mathbf{v}/dt = (d/dt)(\dot{r}\,\boldsymbol{u_r} + r\dot{\theta}\,\boldsymbol{u_\theta})$$

After manipulation, the acceleration can be expressed as

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\boldsymbol{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\boldsymbol{u_\theta}$$



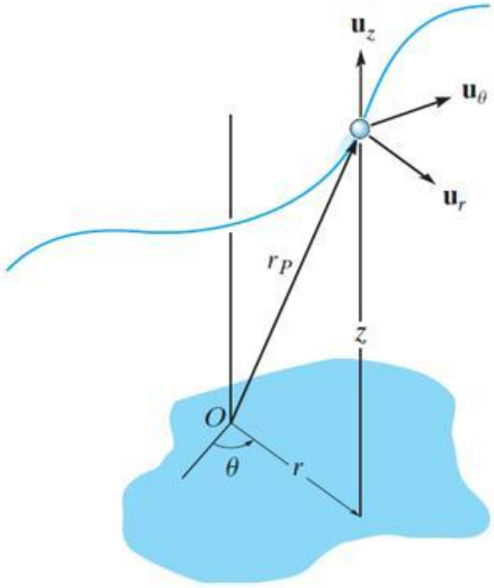
Acceleration

The term $(\ddot{r} - r\dot{\theta}^2)$ is the radial acceleration or a_r .

The term $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the transverse acceleration or a_θ .

The magnitude of acceleration is $a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$

Cylindrical Coordinates



If the particle P moves along a space curve, its position can be written as

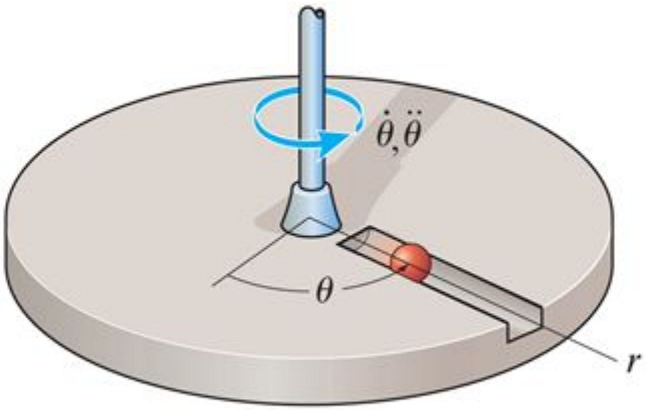
$$\boldsymbol{r_P} = r\boldsymbol{u_r} + z\boldsymbol{u_z}$$

Taking time derivatives and using the chain rule:

Velocity: $\boldsymbol{v_P} = \dot{r}\boldsymbol{u_r} + r\dot{\theta}\boldsymbol{u_\theta} + \dot{z}\boldsymbol{u_z}$

Acceleration: $\boldsymbol{a_P} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u_\theta} + \ddot{z}\boldsymbol{u_z}$

Example



Given: The platform is rotating such that, at any instant, its angular position is $\theta = (4t^{3/2})$ rad, where t is in seconds.

A ball rolls outward so that its position is $r = (0.1t^3)$ m.

Find: The magnitude of velocity and acceleration of the ball when $t = 1.5$ s.

Plan: Use the polar coordinate system.

Solution

$r = 0.1t^3$, $\dot{r} = 0.3t^2$, $\ddot{r} = 0.6t$

$\theta = 4\ t^{3/2}$, $\dot{\theta} = 6\ t^{1/2}$, $\ddot{\theta} = 3\ t^{-1/2}$

At t=1.5 s

$r = 0.3375$ m, $\dot{r} = 0.675$ m/s, $\ddot{r} = 0.9$ m/s^2

$\theta = 7.348$ rad, $\dot{\theta} = 7.348$ rad/s, $\ddot{\theta} = 2.449$ rad/s^2

Substitute into the equation for velocity

$$\mathbf{v} = \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta = 0.675\,\mathbf{u}_r + 0.3375\,(7.348)\,\mathbf{u}_\theta$$

$$= 0.675\,\mathbf{u}_r + 2.480\,\mathbf{u}_\theta$$

$$v = \sqrt{(0.675)^2 + (2.480)^2} = 2.57\text{ m/s}$$

Solution

Substitute in the equation for acceleration:

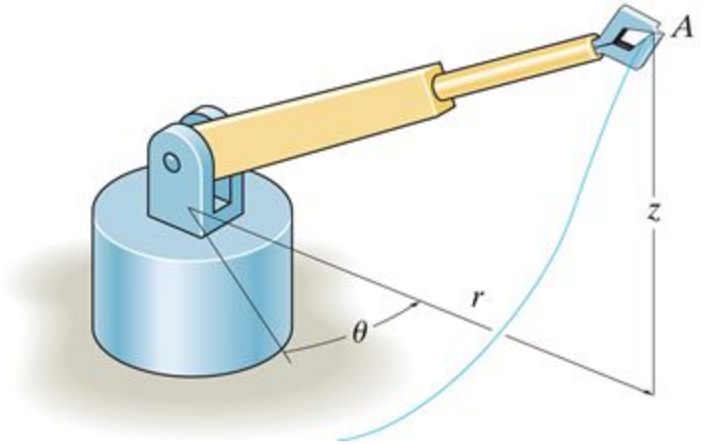
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u_\theta}$$

$$\mathbf{a} = [0.9 - 0.3375(7.348)^2]\,\boldsymbol{u_r} + [0.3375(2.449) + 2(0.675)(7.348)]\,\boldsymbol{u_\theta}$$

$$\mathbf{a} = -17.33\,\boldsymbol{u_r} + 10.75\,\boldsymbol{u_\theta}\ \text{m/s}^2$$

$$a = \sqrt{(-17.33)^2 + (10.75)^2} = 20.4\ \text{m/s}^2$$

Example



Given: The arm of the robot is extending at a constant rate $\dot{r}=1.5$ m/s. At the moment we look at the robot (t=3 s) r=3 m, $z=4t^2$ and $\theta=1.5t$ rad.

Find: The velocity and acceleration of the grip A at t=3 s.

Plan: Use cylindrical coordinates.

Solution

When t = 3 s, r = 3 m and the arm is extending at a constant rate $\dot{r}$ = 1.5 m/s. Thus $\ddot{r}$ = 0 m/s^2.

θ = 1.5t = 4.5 rad, $\dot{\theta}$ = 1.5 rad/s, $\ddot{\theta}$ = 0 rad/s^2.

z = 4t^2 = 36 m, $\dot{z}$ = 8t = 24 m/s, $\ddot{z}$ = 8 m/s^2

Substitute in the equation for velocity

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{u}_r + r\dot{\theta}\,\boldsymbol{u}_\theta + \dot{z}\,\boldsymbol{u}_z$$
$$= 1.5\boldsymbol{u}_r + 3{*}1.5\,\boldsymbol{u}_\theta + 24\,\boldsymbol{u}_z$$
$$= 1.5\boldsymbol{u}_r + 4.5\,\boldsymbol{u}_\theta + 24\,\boldsymbol{u}_z$$

Magnitude v = $\sqrt{1.5^2+4.5^2+24^2}$ = 24.5 m/s

Solution

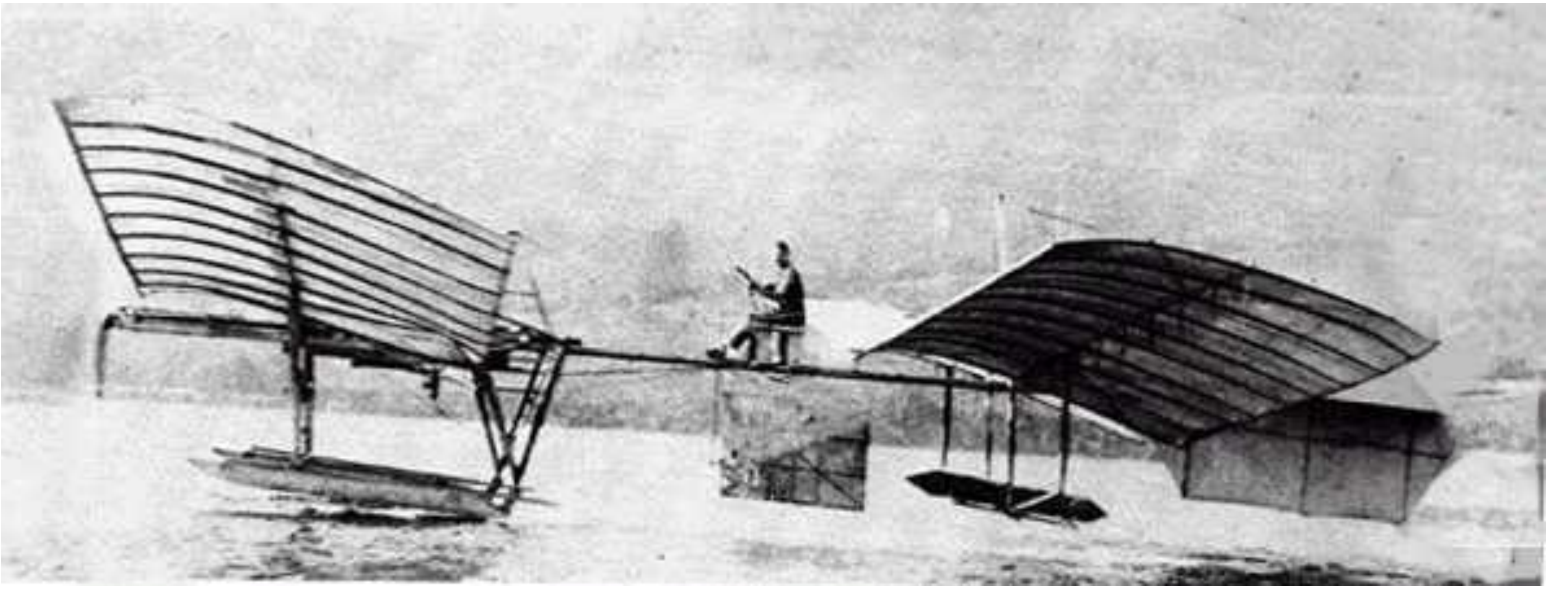
Acceleration equation in cylindrical coordinates

$$
\begin{aligned}
\boldsymbol{a} &= (\ddot{r} - r\dot{\theta}^2)\,\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\boldsymbol{u}_\theta + \ddot{z}\boldsymbol{u}_z \\
&= (0 - 3*1.5^2)\boldsymbol{u}_r + (3*0+2*1.5*1.5)\,\boldsymbol{u}_\theta + 8\,\boldsymbol{u}_z \\
&= -6.75\,\boldsymbol{u}_r + 4.5\,\boldsymbol{u}_\theta + 8\,\boldsymbol{u}_z
\end{aligned}
$$

Magnitude $v = \sqrt{(-6.75)^2+4.5^2+8^2} = 11.4$ m/s

Ugly aircraft competition

Curtis Aerodrome (1914)



1. 1
2. 2
3. 3
4. 4
5. 5
6. 6
7. 7
8. 8
9. 9
10. 10

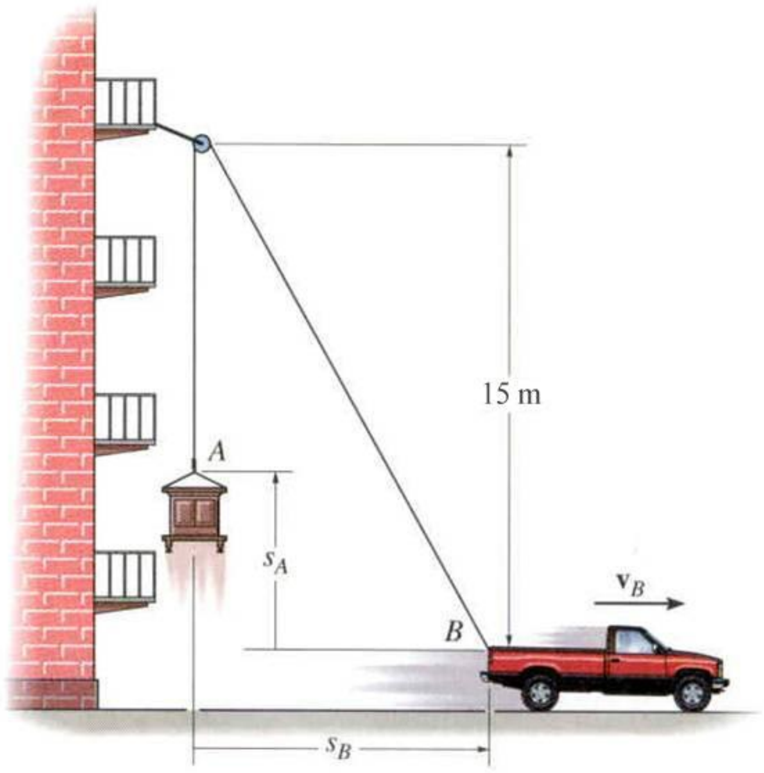
Chapter 12: Kinematics of a Particle

Section 12.9: Absolute Dependent Motion of Two Particles

Learning Objective

Be able to relate the positions, velocities, and accelerations of particles undergoing dependent motion.

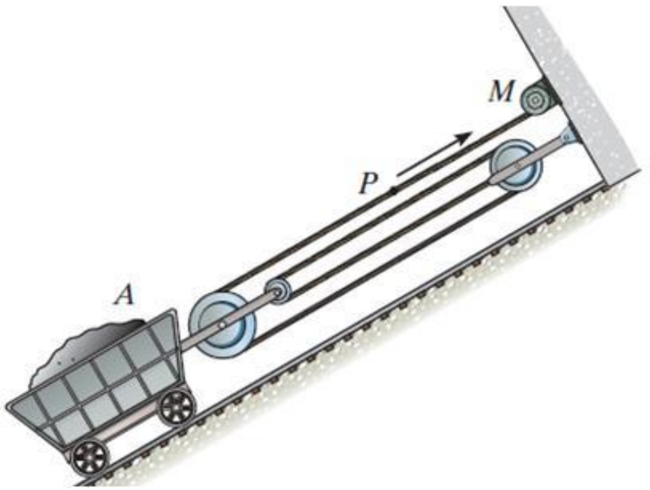
Applications



Rope and pulley arrangements are often used to assist in lifting heavy objects. The total lifting force required from the truck depends on both the weight and the acceleration of the cabinet.

How can we determine the acceleration and velocity of the cabinet if the acceleration of the truck is known?

Applicatons



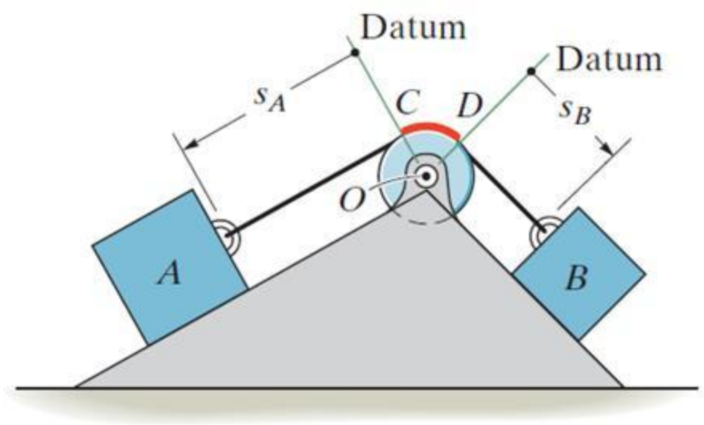
The cable and pulley system shown can be used to modify the speed of the mine car, A, relative to the speed of the motor, M.

It is important to establish the relationships between the various motions in order to determine the power requirements for the motor and the tension in the cable.

For instance, if the speed of the cable (P) is known because we know the motor characteristics, how can we determine the speed of the mine car? Will the slope of the track have any impact on the answer?

Dependent Motion

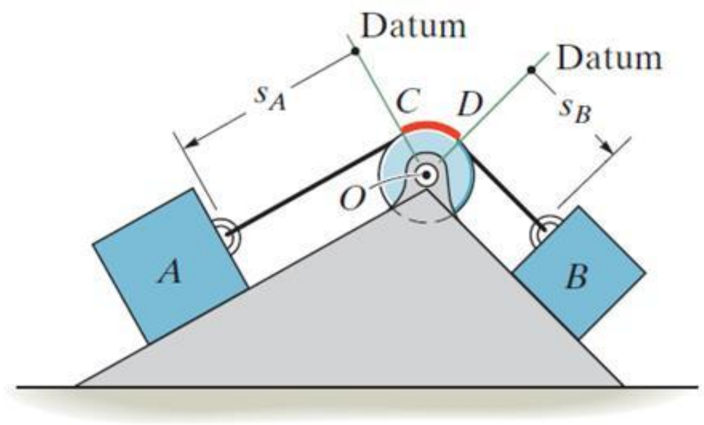
In many kinematics problems, the motion of one object will depend on the motion of another object.



The blocks in this figure are connected by an inextensible cord wrapped around a pulley. If block A moves downward along the inclined plane, block B will move up the other incline.

The motion of each block can be related mathematically by defining position coordinates, s_A and s_B. Each coordinate axis is defined from a fixed point or datum line, measured positive along each plane in the direction of motion of each block.

Dependent Motion



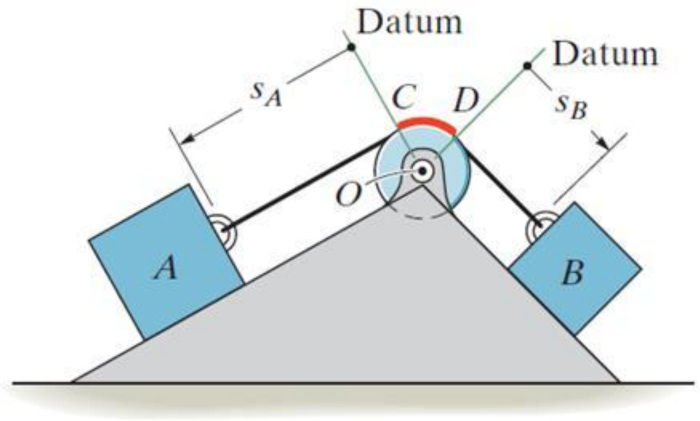
In this example, position coordinates s_A and s_B can be defined from fixed datum lines extending from the center of the pulley along each incline to blocks A and B.

If the cord has a fixed length, the position coordinates s_A and s_B are related mathematically by the equation

$$s_A + l_{CD} + s_B = l_T$$

Here l_T is the total cord length and l_{CD} is the length of cord passing over the arc CD on the pulley.

Dependent Motion



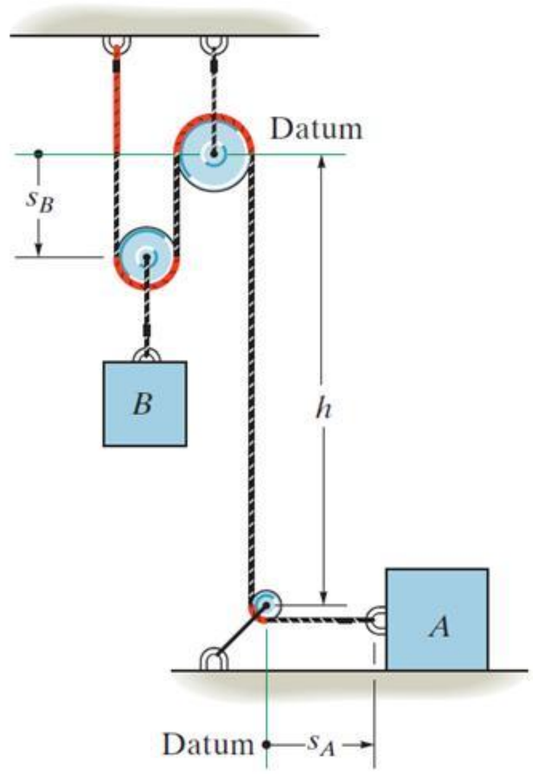
The velocities of blocks A and B can be related by differentiating the position equation. Note that l_{CD} and l_T remain constant,
so $dl_{CD}/dt = dl_T/dt = 0$

$$ds_A/dt + ds_B/dt = 0 \quad \Rightarrow \quad v_B = -v_A$$

The negative sign indicates that as A moves down the incline (positive s_A direction), B moves up the incline (negative s_B direction).

Accelerations can be found by differentiating the velocity expression. Prove to yourself that $a_B = -a_A$.

Example

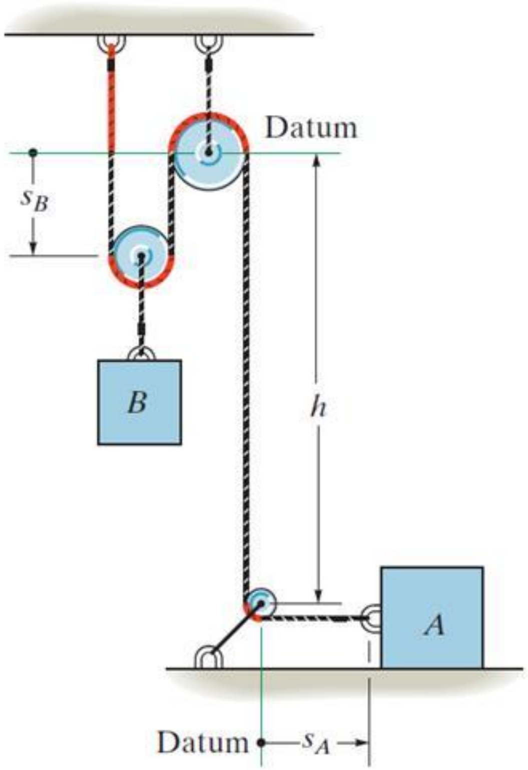


Consider a more complicated example. Position coordinates (s_A and s_B) are defined from fixed datum lines, measured along the direction of motion of each block.

Note that s_B is only defined to the center of the pulley above block B, since this block moves with the pulley. Also, *h* is a constant.

The red colored segments of the cord remain constant in length during motion of the blocks.

Solution



The position coordinates are related by the equation

$2s_B + h + s_A = l_T$

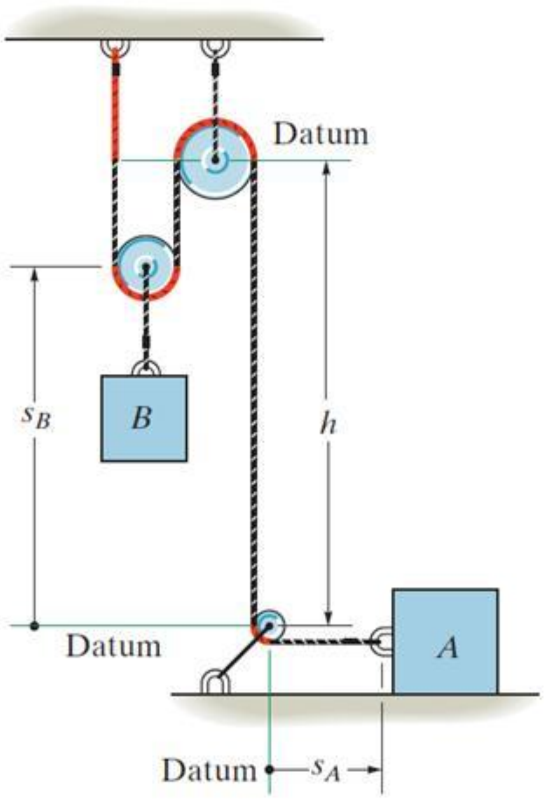
Where l_T is the total cord length minus the lengths of the red segments.

Since l_T and *h* remain constant during the motion, the velocities and accelerations can be related by two successive time derivatives:

$2v_B = -v_A$ and $2a_B = -a_A$

When block B moves downward ($+s_B$), block A moves to the left ($-s_A$). Remember to be consistent with your sign convention!

Solution



This example can also be worked by defining the position coordinate for B (s_B) <u>from the bottom pulley</u> instead of the top pulley.

The position, velocity, and acceleration relations then become

$2(h - s_B) + h + s_A = l_T$

and $\quad 2v_B = v_A \qquad 2a_B = a_A$

Prove to yourself that the results are the same, even if the sign conventions are different than the previous formulation.

Dependent Motion: Procedure

These procedures can be used to relate the dependent motion of particles moving along rectilinear paths (only the magnitudes of velocity and acceleration change, not their line of direction).

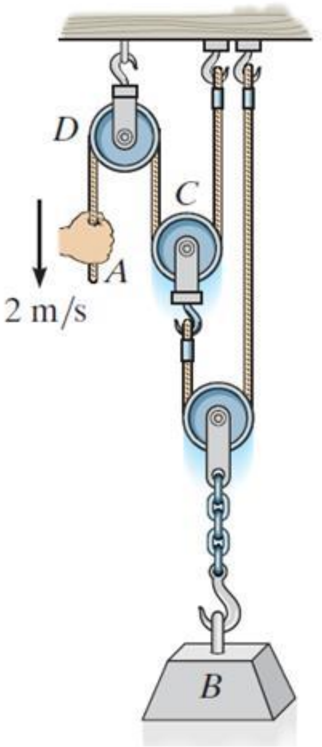
1. Define position coordinates from **fixed datum lines**, along the path of each particle. Different datum lines can be used for each particle.

2. Relate the position coordinates to the cord length. Segments of cord that do not change in length during the motion may be left out.

3. If a system contains more than one cord, relate the position of a point on one cord to a point on another cord. Separate equations are written for each cord.

4. Differentiate the position coordinate equation(s) to relate velocities and accelerations. Keep track of signs!

Example



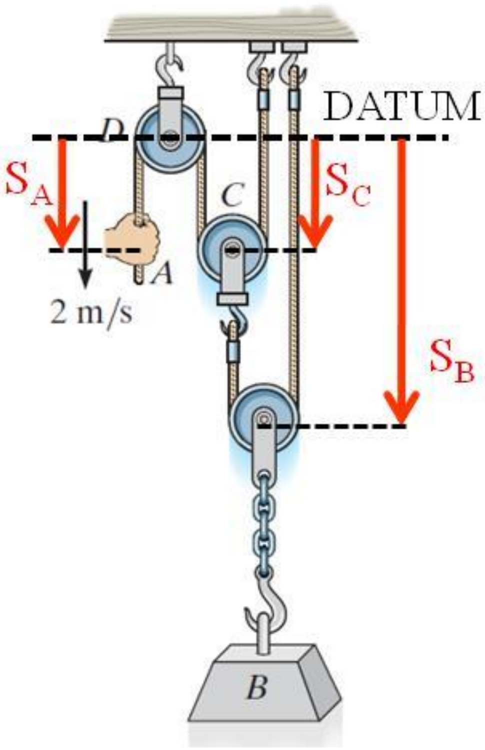
Given: In the figure on the left, the cord at A is pulled down with a speed of 2 m/s.

Find: The speed of block B.

Plan: There are two cords involved in the motion in this example. There will be two position equations (one for each cord). Write these two equations, combine them, and then differentiate them.

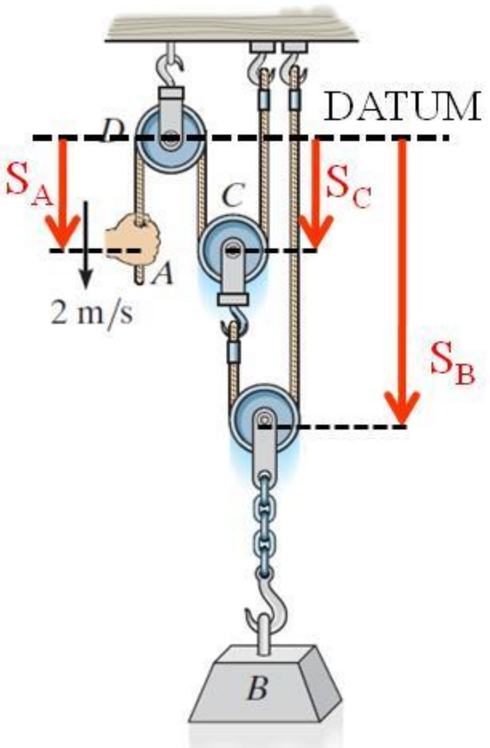
Solution

1) Define the position coordinates from a fixed datum line. Three coordinates must be defined: one for point A (s_A), one for block B (s_B), and one for block C (s_C).



- Define the datum line through the top pulley (which has a fixed position).
- s_A can be defined to the point A.
- s_B can be defined to the center of the pulley above B.
- s_C is defined to the center of pulley C.
- All coordinates are defined as positive down and along the direction of motion of each point/object.

Solution



2) Write position/length equations for each cord. Define l_1 as the length of the first cord, minus any segments of constant length. Define l_2 in a similar manner for the second cord:

$$\text{Cord 1: } s_A + 2s_C = l_1$$
$$\text{Cord 2: } s_B + (s_B - s_C) = l_2$$

3) Eliminating s_C between the two equations, we get:

$$s_A + 4s_B = l_1 + 2l_2$$

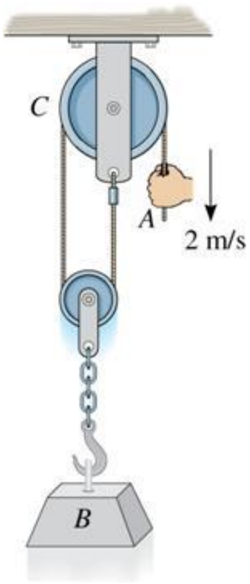
4) Relate velocities by differentiating this expression. Note that l_1 and l_2 are constant lengths.

$$v_A + 4v_B = 0 \quad \Rightarrow \quad v_B = -0.25v_A = -0.25(2) = -0.5 \text{ m/s}$$

The velocity of block B is 0.5 m/s up (negative s_B direction).

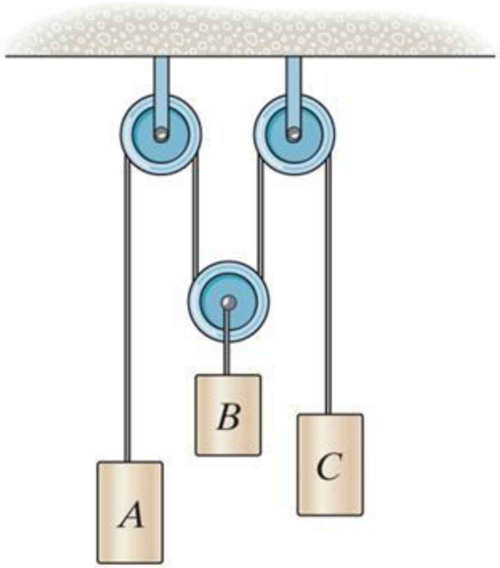
Quiz

Determine the speed of block B.



1. 1 m/s
2. 2 m/s
3. 4 m/s
4. None of the above.

Example



Given: In this pulley system, block A is moving downward with a speed of 4 m/s while block C is moving up at 2 m/s.

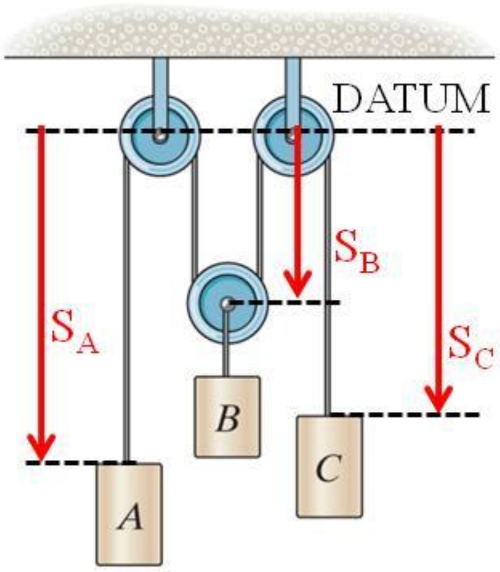
Find: The speed of block B.

Plan:

All blocks are connected to a single cable, so only one position/length equation will be required. Define position coordinates for each block, write out the position relation, and then differentiate it to relate the velocities.

Solution

1) A datum line can be drawn through the upper, fixed, pulleys and position coordinates defined from this line to each block (or the pulley above the block).



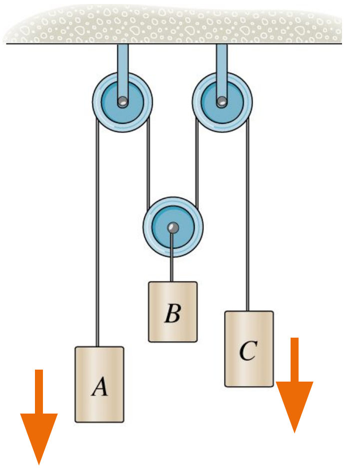
2) Defining s_A, s_B, and s_C as shown, the position relation can be written:
$s_A + 2s_B + s_C = l$

3) Differentiate to relate velocities:
$v_A + 2v_B + v_C = 0$
$\Rightarrow \quad 4 + 2v_B + (-2) = 0$
$\Rightarrow \quad v_B = -1$ m/s

The velocity of block B is 1 m/s up (negative s_B direction).

Quiz

Determine the speed of block B when block A is moving down at 6 m/s while block C is moving down at 18 m/s.

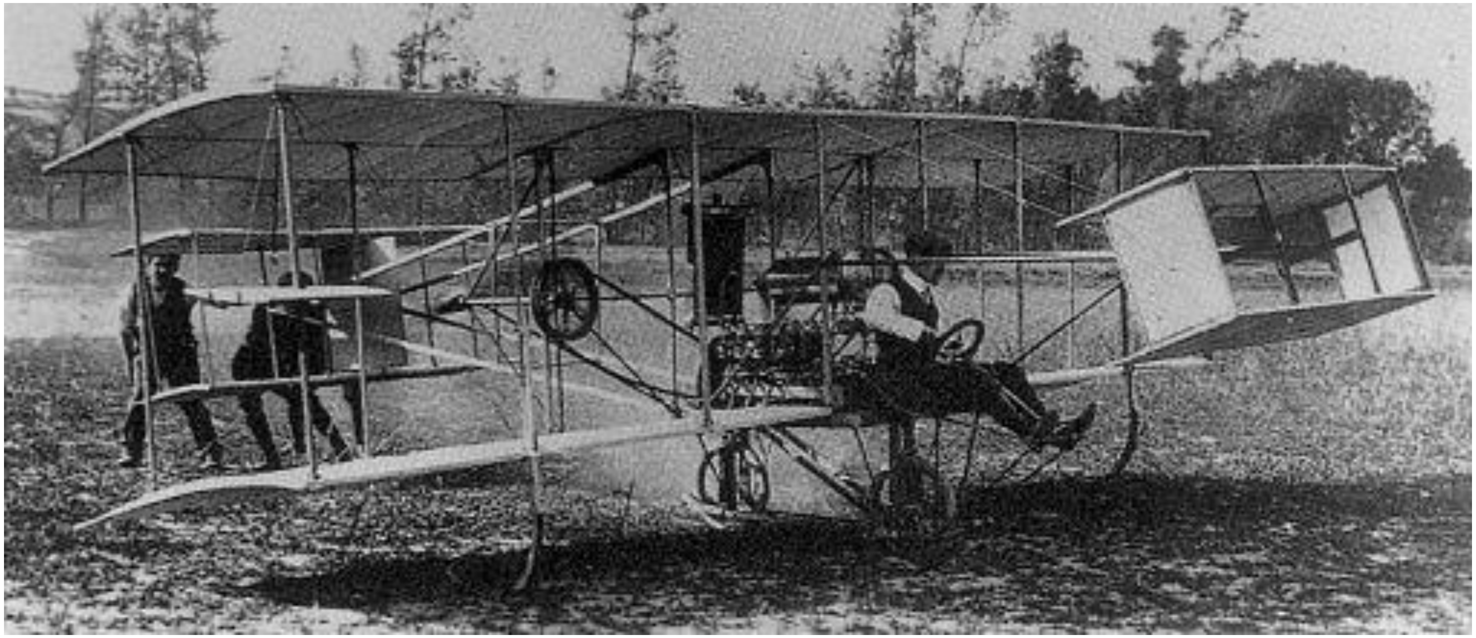


v_A=6 m/s v_C=18 m/s

1. 24 m/s
2. 3 m/s
3. 12 m/s
4. 9 m/s

Ugly aircraft competition

Koechlin biplane (1908)



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2. 2
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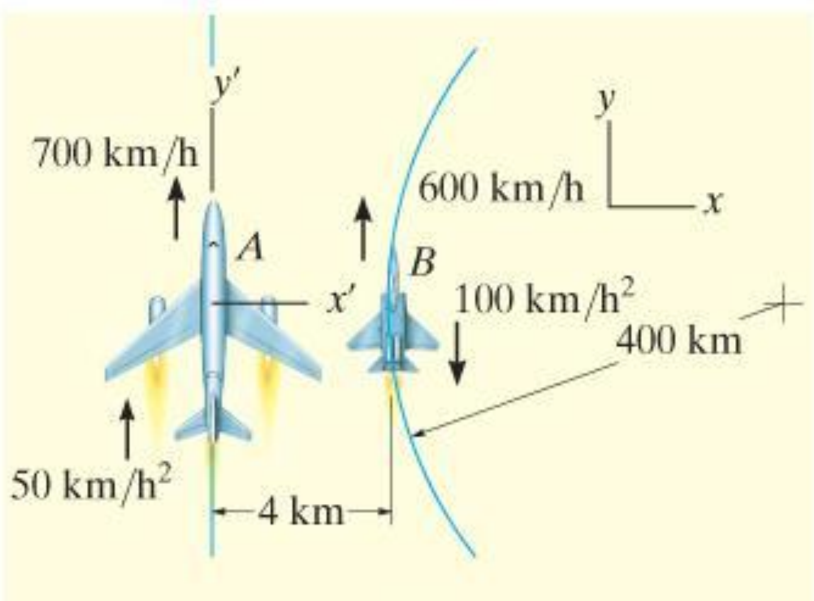
Chapter 12: Kinematics of a Particle

Section 12.10: Relative Motion of Two Particles Using Translating Axes

Learning Objective

Be able to relate the positions, velocities, and accelerations of particles undergoing relative motion.

Applications

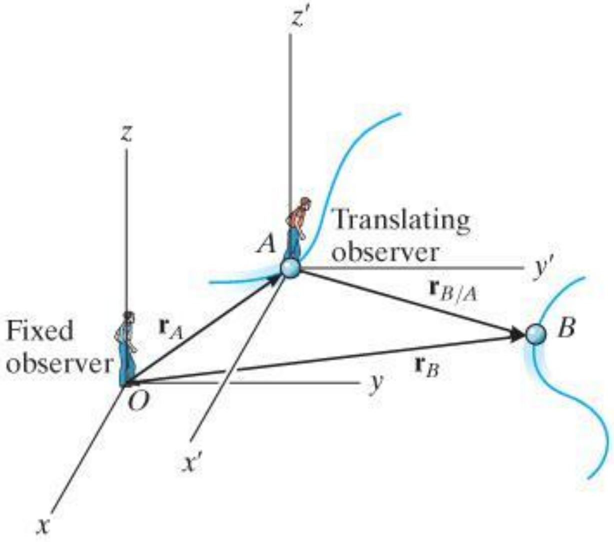


(a)

A fighter aircraft is trying to intercept an airliner because communication got lost.

The fighter pilot needs to make sure he does intercept the airliner at the correct location at the correct altitude

Relative Motion: Position



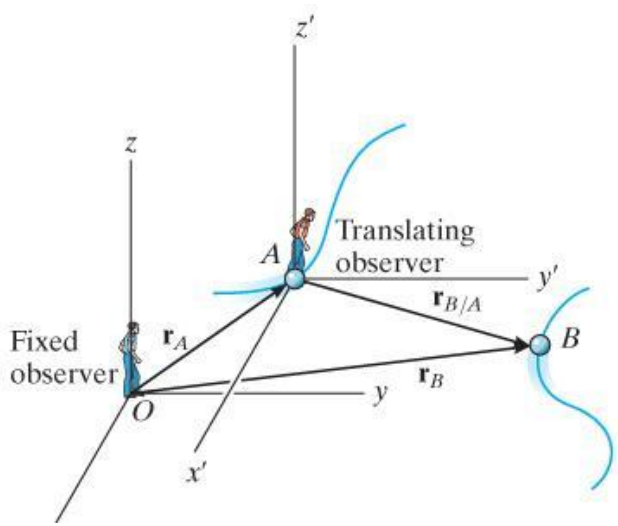
Particles A and B are moving both along their own path. Their absolute position vectors are $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ when measured from the position of the fixed observer.

There is a second reference frame x′-y′-z′ that is moving with respect to the observer at O but is fixed to particle A. This reference frame is **only allowed to translate** with respect to the fixed reference frame.

The position of B can be measured relative to A with the relative-position vector $\boldsymbol{r}_{B/A}$. The following relation holds:

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$

Relative Motion: Velocity and Acceleration



For the velocity one can write:

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

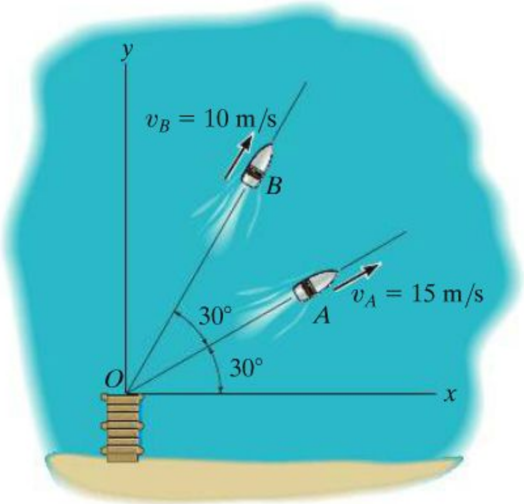
And for the acceleration:

$$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}$$

Relative Motion: Procedure

1. First specify the particle A that is the origin for the translating x′, y′, z′-axes. Usually this point has a known velocity and/or acceleration.
2. Since vector addition forms triangles there can be at most two unknowns. The represent magnitudes and/or directions of the vector quantities.
3. These unknowns can be solved for either graphically or numerically using trigonometry or by resolving each of the three vectors into a coordinate system and thereby generating a set of scalars.

Example



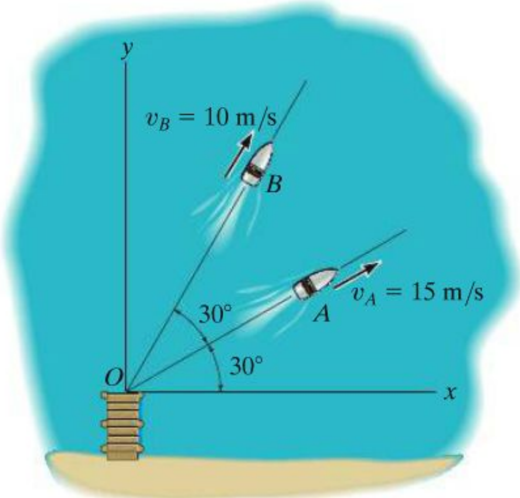
Given: Two boats are leaving the pier at the same moment but with different speeds and in different directions.

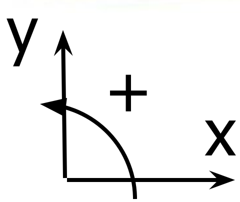
Find: What is the distance between them after 4 seconds and what is the direction of boat B with respect to boat A?

Plan:

The origin of the x- and y-axes are located at O. First determine the positions of A and B after 4 seconds. Then use relative positions to find the position of B with respect to A. Use vectors!!

Example





$\boldsymbol{v}_A = 15 \cos 30° \, \boldsymbol{i} + 15 \sin 30° \, \boldsymbol{j}$

$\boldsymbol{v}_B = 10 \cos 60° \, \boldsymbol{i} + 10 \sin 60° \, \boldsymbol{j}$

After 4 seconds

$\boldsymbol{r}_A = 60 \cos 30° \, \boldsymbol{i} + 60 \sin 30° \, \boldsymbol{j}$

$\boldsymbol{r}_B = 40 \cos 60° \, \boldsymbol{i} + 40 \sin 60° \, \boldsymbol{j}$

Then use $\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$

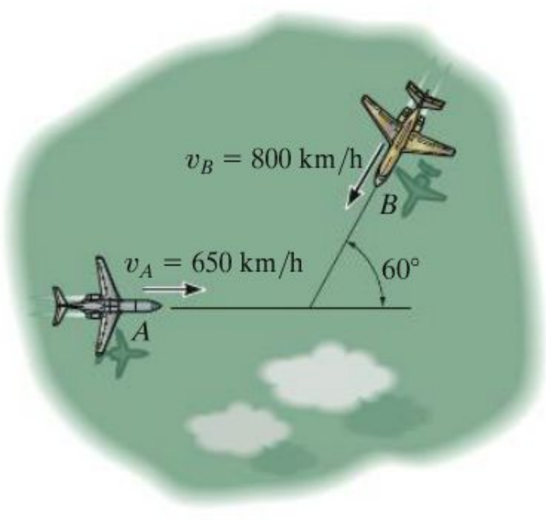
Thus $\boldsymbol{r}_{B/A} = \boldsymbol{r}_B - \boldsymbol{r}_A = (40 \cos 60^\circ - 60 \cos 30^\circ) \, \boldsymbol{i} + (40 \sin 60^\circ - 60 \sin 30^\circ) \, \boldsymbol{j}$

$= -31.96 \, \boldsymbol{i} + 4.64 \, \boldsymbol{j}$

Distance $= \sqrt{31.96^2 + 4.64^2} = 32.3$ m   angle 171.7°

Quiz

Two planes A and B are flying at constant speed. Determine the magnitude of the velocity of plane B relative to plane A



1. 693 km/h
2. 650 km/h
3. 400 km/h
4. 1258 km/h

Example

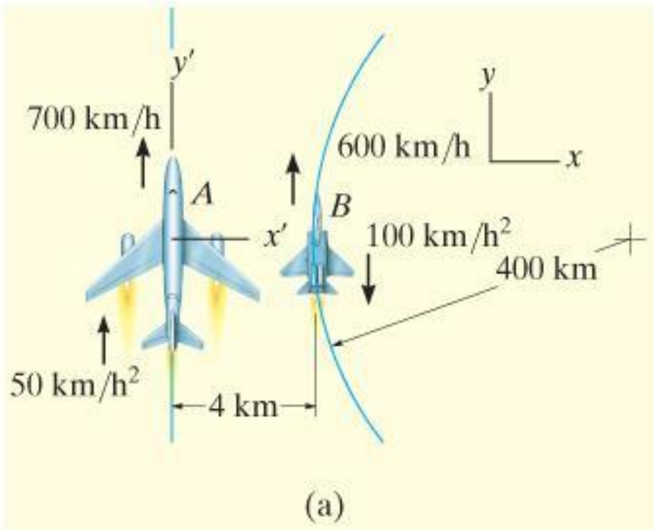


Figure: 12_044a_EX026

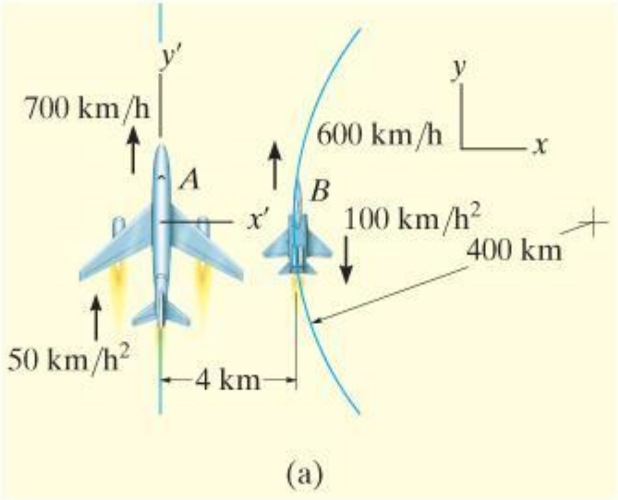
Given: Aircraft A is flying along in a straight line, whereas fighter B is flying along a circular path with a radius of curvature of 400 km.

Find: Determine the velocity and acceleration of fighter B as measured by the pilot of aircraft A.

Plan:

The origin of the x- and y-axes are located in an arbitrary but fixed point. The translating reference frame is attached to A. Then apply the relative velocity and relative acceleration in scalar form. This is done because at the instant of measuring both reference frames are parallel.

Solution

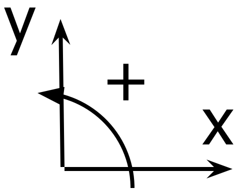


1) For the velocity one can write:

$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$

$600\ \boldsymbol{j}$ km/h = $700\ \boldsymbol{j}$ km/h + $v_{B/A}$

$v_{B/A} = -100\ \boldsymbol{j}$ km/h   angle 270°



2) For the acceleration: Fighter B has both normal and tangential accelerations since it is flying a curved path.

$(a_B)_t = -100\ \boldsymbol{j}$ km/h^2

$(a_B)_n = v_B^{\,2}/\rho = (600 \text{ km/h})^2/400 \text{ km} = 900\boldsymbol{i}$ km/h^2

Thereby:

$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A} \Rightarrow 900\boldsymbol{i} - 100\boldsymbol{j} = 50\boldsymbol{j} + \boldsymbol{a}_{B/A}$

Thus:

$\boldsymbol{a}_{B/A} = (900\boldsymbol{i} - 150\boldsymbol{j})$ km/h^2 angle = 350.5°