Solving linear recurrence relations

Irina Prosvirnina

- Linear homogeneous recurrence relations with constant coefficients
- Solving linear homogeneous recurrence relations with constant coefficients
- Solving linear homogeneous recurrence relations with constant coefficients of degree two and of degree three
- Linear nonhomogeneous recurrence relations with constant coefficients

Linear homogeneous recurrence relations with constant coefficients

Definition 1

A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$.

Linear homogeneous recurrence relations with constant coefficients

- $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

A sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions:

$$\boldsymbol{a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}}.$$

Linear homogeneous recurrence relations with constant coefficients

- Example 1

The recurrence relation

$$P_n = (1.11)P_{n-1}$$

is a linear homogeneous recurrence relation of degree one.

Linear homogeneous recurrence relations with constant coefficients

<u>Example 2</u>

The recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

is a linear homogeneous recurrence relation of degree two.

The sequence of **Fibonacci numbers** satisfies this recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0, f_1 = 1$.

Linear homogeneous recurrence relations with constant coefficients

<u>Example 3</u>

The recurrence relation

$$a_n = a_{n-5}$$

is a linear homogeneous recurrence relation of degree five.

Linear homogeneous recurrence relations with constant coefficients

Example 4

The recurrence relation

$$a_n = a_{n-1} + a_{n-2}^2$$

is not linear.

Linear homogeneous recurrence relations with constant coefficients

Example 5

The recurrence relation

$$H_n = 2H_{n-1} + 1$$

is not homogeneous.

Linear homogeneous recurrence relations with constant coefficients

Example 6

The recurrence relation

$$B_n = nB_{n-1}$$

does not have constant coefficients.

Solving linear homogeneous recurrence relations with constant coefficients

- The basic approach for solving linear homogeneous recurrence relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is to look for solutions of the form

$$\boldsymbol{a_n = r^n},$$

where r is a constant.

Solving linear homogeneous recurrence relations with constant coefficients

- Note that

$$a_n = r^n$$

is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

Solving linear homogeneous recurrence relations with constant coefficients

- $$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

When both sides of this equation are divided by r^{n-k}, we obtain the equation

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_{k-1} r + c_k.$$

When the right-hand side is subtracted from the left we obtain the equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

Solving linear homogeneous recurrence relations with constant coefficients

- Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution of linear homogeneous recurrence relations with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad (*)$$

is a solution **if and only if**

r is a solution of this last equation

$$\boldsymbol{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0}.$$

We call this the **characteristic equation** of the recurrence relation $(*)$.

The solutions of this equation are called the **characteristic roots** of the recurrence relation $(*)$.

Solving linear homogeneous recurrence relations with constant coefficients

As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

- Theorem 1

Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if $\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof of theorem 1

- If r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, α_1 and α_2 are constants then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$.

$$r_1^2 = c_1 r_1 + c_2, \quad r_2^2 = c_1 r_2 + c_2$$
$$\Downarrow$$

$$
\begin{aligned}
& c_1 a_{n-1} + c_2 a_{n-2} = \\
& c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) = \\
& \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) = \\
& \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 = \\
& \alpha_1 r_1^n + \alpha_2 r_2^n = \\
& a_n \blacksquare
\end{aligned}
$$

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

Let $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

and the initial conditions $a_0 = C_0, a_1 = C_1$ hold.

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

It will be shown that there are constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions $a_0 = C_0, a_1 = C_1$.

Proof of theorem 1

2 If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

This requires that

$$a_0 = C_0 = \alpha_1 + \alpha_2,$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2,$$

We can solve these two equations for α_1 and α_2:

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}.$$

Proof of theorem 1

- If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

Hence, with these values for

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}, \alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, satisfies the two initial conditions $a_0 = C_0, a_1 = C_1$.

Proof of theorem 1

? If the sequence $\{a_n\}$ is a solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, where r_1 and r_2 are distinct roots of $r^2 - c_1 r - c_2 = 0$.

We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$.

Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$. ■

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 7

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution

The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Find its roots.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 7

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution

The characteristic equation of the recurrence relation is

$$r^2 - r - 2 = 0.$$

Its roots are

$$\mathbf{r = 2 \text{ and } r = -1.}$$

By theorem 1, $a_n = \alpha_1 2^n + \alpha_2(-1)^n$.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 7

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$

$a_0 = 2 = \alpha_1 + \alpha_2,$

$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1), \Rightarrow$

$\alpha_1 = 3, \alpha_2 = -1, \Rightarrow$

$a_n = 3 \cdot 2^n - (-1)^n.$

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

- Example 8 (Fibonacci numbers)

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0 = 0$ and $f_1 = 1$?

Solution

The characteristic equation of the recurrence relation is

$$r^2 - r - 1 = 0.$$

Its roots are

$$r = (1 + \sqrt{5})/2 \text{ and } r = (1 - \sqrt{5})/2.$$

By theorem 1, $f_n = \alpha_1\left((1 + \sqrt{5})/2\right)^n + \alpha_2\left((1 - \sqrt{5})/2\right)^n$.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 8 (Fibonacci numbers)

What is the solution of the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

with $f_0 = 0$ and $f_1 = 1$?

Solution

$$f_n = \alpha_1\left((1+\sqrt{5})/2\right)^n + \alpha_2\left((1-\sqrt{5})/2\right)^n$$

$f_0 = 0 = \alpha_1 + \alpha_2,$

$f_1 = 1 = \alpha_1\left((1+\sqrt{5})/2\right) + \alpha_2\left((1-\sqrt{5})/2\right), \Rightarrow$

$\alpha_1 = 1/\sqrt{5}, \alpha_2 = -1/\sqrt{5}, \Rightarrow$

$f_n = 1/\sqrt{5}\left((1+\sqrt{5})/2\right)^n + (-1/\sqrt{5})\left((1-\sqrt{5})/2\right)^n.$

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

- <u>Theorem 2</u>

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has only one root r_0.

A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if $\boldsymbol{a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n}$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 9

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$?

Solution

The characteristic equation of the recurrence relation is

$$r^2 - 6r + 9 = 0.$$

Its root is

$$r = 3.$$

By theorem 2, $a_n = \alpha_1 3^n + \alpha_2 n 3^n$.

Solving linear homogeneous recurrence relations with constant coefficients of **degree two**

Example 9

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$?

Solution

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

$a_0 = 1 = \alpha_1,$

$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3, \Rightarrow$

$\alpha_1 = 1, \alpha_2 = 1, \Rightarrow$

$a_n = 3^n + n3^n.$

Solving linear homogeneous recurrence relations with constant coefficients of **degree three**

Theorem 3

Let $c_1, c_2, \ldots, c_k$ be real numbers.

Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$.

A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if $\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n}$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Solving linear homogeneous recurrence relations with constant coefficients of **degree three**

Example 10

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2, a_1 = 5, a_2 = 15.$

Solution

The characteristic equation of the recurrence relation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Find its roots.

Solving linear homogeneous recurrence relations with constant coefficients of **degree three**

Example 10

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2, a_1 = 5, a_2 = 15.$

Solution

The characteristic equation of the recurrence relation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Its roots are

$$\boldsymbol{r_1 = 1, r_2 = 2, r_3 = 3.}$$

By theorem 3, $a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$.

Solving linear homogeneous recurrence relations with constant coefficients of **degree three**

Example 10

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2, a_1 = 5, a_2 = 15.$

Solution

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$$

$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$

$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$

$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9, \Rightarrow$

$\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2, \Rightarrow$

$a_n = 1 - 2^n + 2 \cdot 3^n.$

Linear nonhomogeneous recurrence relations with constant coefficients

Definition 2

A linear nonhomogeneous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, $c_k \neq 0$; $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the **associated homogeneous recurrence relation**.

It plays an important role in the solution of the nonhomogeneous recurrence relation.

Linear nonhomogeneous recurrence relations with constant coefficients

Example 11

The recurrence relation

$$a_n = a_{n-1} + 2^n$$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = a_{n-1}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

<u>Example 12</u>

The recurrence relation

$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = a_{n-1} + a_{n-2}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 13

The recurrence relation

$$a_n = 3a_{n-1} + n3^n$$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = 3a_{n-1}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

- Example 14

The recurrence relation

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

is a linear nonhomogeneous recurrence relation with constant coefficients.

The associated linear homogeneous recurrence relation is

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Theorem 4

If $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$, where $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Proof of theorem 4

- Because $\left\{a_n^{(p)}\right\}$ is a particular solution of the nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Proof of theorem 4

- Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Proof of theorem 4

- So,

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n),$$
$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} =$$
$$= c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\left\{b_n - a_n^{(p)}\right\}$ is a solution of the associated homogeneous linear recurrence relation, say, $\left\{a_n^{(h)}\right\}$.

Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ ■

Linear nonhomogeneous recurrence relations with constant coefficients

<u>Example 15</u>

Find all solutions of the recurrence relation

$$a_n = 3a_{n-1} + 2n.$$

<u>Solution</u>

This is a linear nonhomogeneous recurrence relation.

The solutions of its associated homogeneous recurrence relation

$$a_n = 3a_{n-1}$$

are $a_n^{(h)} = \alpha \cdot 3^n$.

Linear nonhomogeneous recurrence relations with constant coefficients

Example 15

Find all solutions of the recurrence relation

$$a_n = 3a_{n-1} + \mathbf{2n}.$$

Solution

We now find a particular solution.

Suppose that $p_n = \boldsymbol{cn + d}$, where c and d are constants, such a solution.

$$cn + d = 3(c(n-1) + d) + 2n,$$
$$(2 + 2c)n + (2d - 3c) = 0.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 15

Find all solutions of the recurrence relation

$$a_n = 3a_{n-1} + \mathbf{2n}.$$

Solution

$(2+2c)n + (2d-3c) = 0,$

$$\begin{cases} 2+2c=0, \\ 2d-3c=0, \end{cases} \Rightarrow$$

$$c=-1, d=-\frac{3}{2}, \Rightarrow$$

$$a_n^{(p)} = -n - 3/2, \Rightarrow$$

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha \cdot 3^n.$$

Linear nonhomogeneous recurrence relations with constant coefficients

<u>Example 16</u>

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

<u>Solution</u>

This is a linear nonhomogeneous recurrence relation.

The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$.

Linear nonhomogeneous recurrence relations with constant coefficients

Example 16

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + \mathbf{7^n}.$$

Solution

We now find a particular solution.

Suppose that $F(n) = \boldsymbol{C \cdot 7^n}$, where C is a constant, such a solution.

$$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n,$$

$$49C = 35C - 6C + 49,$$

$$C = 49/20.$$

Linear nonhomogeneous recurrence relations with constant coefficients

Example 16

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + \mathbf{7^n}.$$

Solution

$a_n^{(p)} = (49/20)7^n, \Rightarrow$

$a_n = a_n^{(p)} + a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$