NUFYP Mathematics

Lecture 2.2 Modelling with Exponentials and Logarithms

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Lecture Outline

- Graphs of transformed Exponential functions
- Graphs of transformed Logarithmic functions
- Mathematical modelling
- Exponential Growth and Decay
- Modelling with Exponential and Logarithmic functions

Introduction

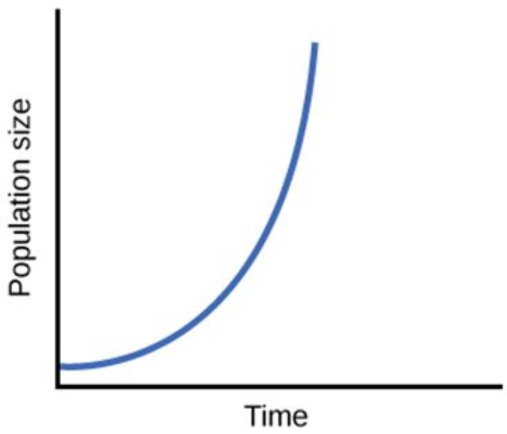
Mathematical models

Modelling using Exponents and Logarithms

Often **data does not fit** to a **linear** or other **polynomial function**. When this happens there are some functions such as **Exponential** and **Logarithmic functions** that are **used to model phenomena occurring** in **nature**.

Mathematical models

1. Exponential Growth

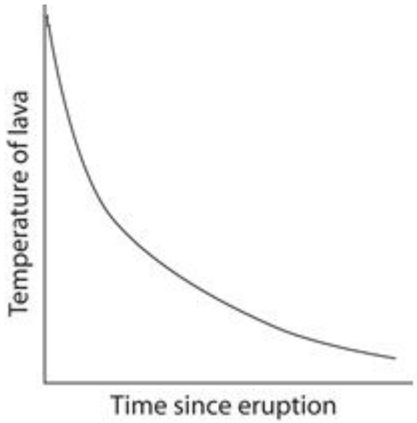


$$y = a * e^{bx}$$

Used to model:

- Population growth
- Compound interest

2. Exponential Decay



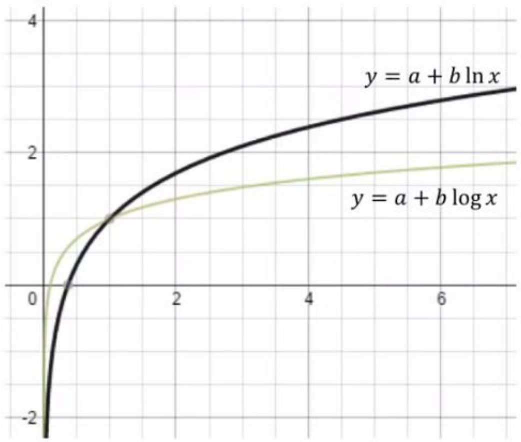
$$y = a * e^{-bx}$$

Used to model:

- Radioactive decay
- Carbon dating

Mathematical models

3. Logarithmic Growth



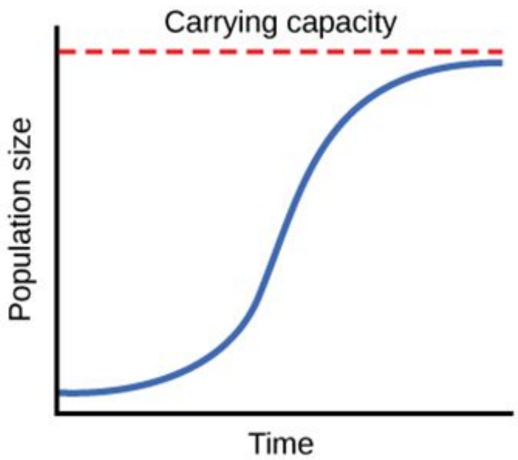
$$y = a + b * ln(x)$$

$$y = a + b * log(x)$$

Used to model:

- Earthquakes
- Sound levels

4. Logistic Growth



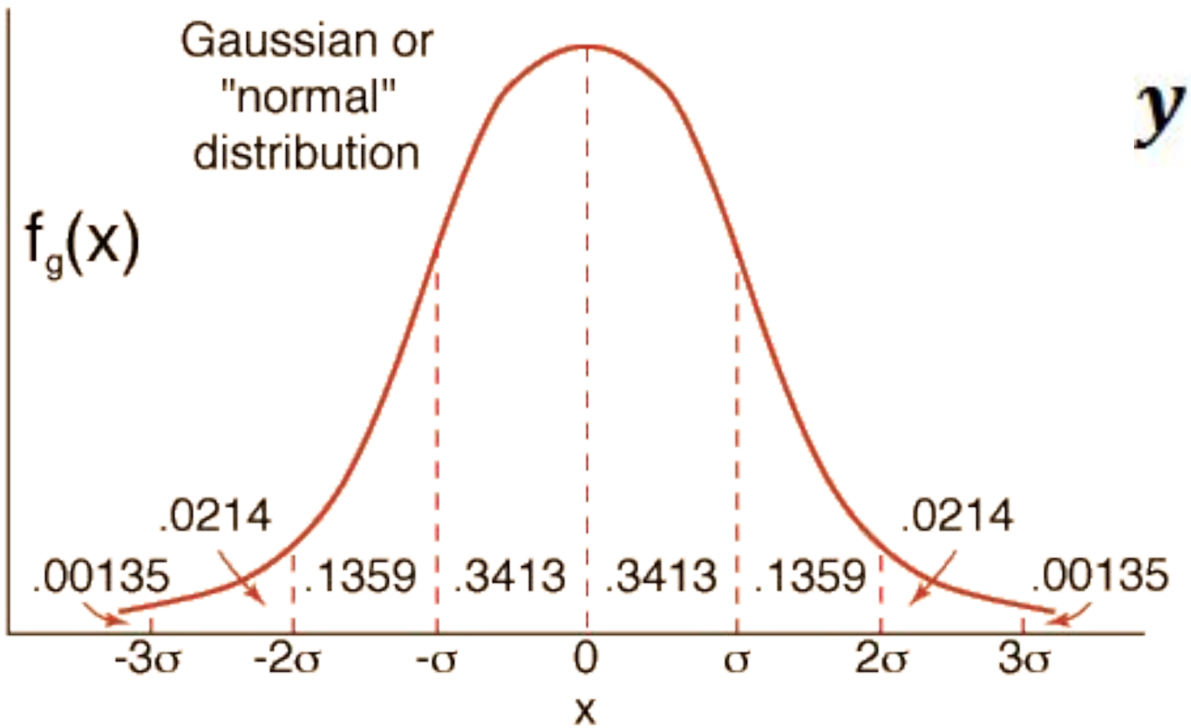
$$y = \frac{a}{1 + be^{-rx}}$$

Used to model:

- Spread of disease
- Learning

Mathematical models

5. Gaussian distribution (Normal distrib.)



Equation:

$$y = a * e^{-(x-b)^2/c}$$

Used to model:

- Probability distribution
- Standardized test (SAT) marks

Mathematical models

1. **Exponential Growth**
2. **Exponential Decay**
3. **Logarithmic Model**

Note: In this lecture we will focus only on these 3 models

4. Logistic Growth
5. Gaussian Distribution (Normal Distribution)

Introduction to "e"

Mathematical constant *e* is a real, irrational and transcendental number approximately equal to:

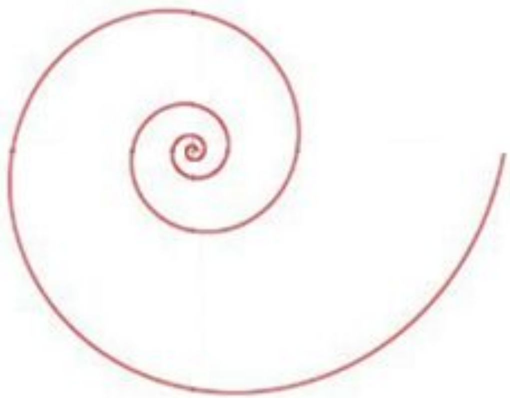
e = 2.71828 18284 59045 23536 02874 71352 66249 77572 47093 6999595749 66967 62772 40766 30353 54759 45713 82178 53516 6427427466 39193 20030 59921 81741 35966 29043 57290 03342 9526059563 07381 32328 62794 34907 63233 82988 07531 95251 01901 …

A **transcendental number** is a number that is not a root of any polynomial with integer coefficients. They are the opposite of algebraic numbers, which are numbers that are roots of some integer polynomial.

Do you know any other transcendental number? Answer: **π**

“e” is almost everywhere

$$r = ke^{a\varphi}$$



The logarithmic spiral is a shape that **appears in nature**, and is found in such places as **shells**, horns, tusks, sunflowers, and even **spiral galaxies**. However, **despite** the **aesthetic wonders** of the **number**, **it was actually first discovered** in a **pragmatic financial investigation** of the **behavior of compound interest**.

2.2.1 Sketch graphs of transformed exponential functions

Let's sketch graphs of transformed **exponential functions** such as:

Horizontal scaling factor, scale factor 1/c

$$y = a + b * e^{cx+d}$$

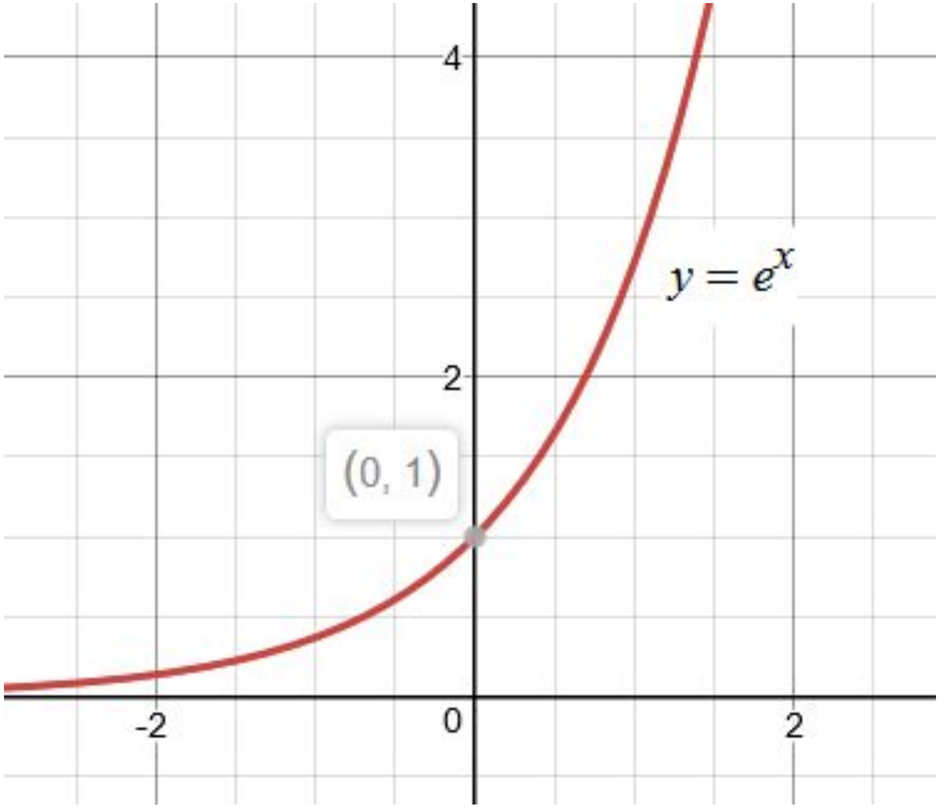
Vertical translation

Horizontal translation

Vertical scaling factor, scale factor b

We assume that ***a***, ***b***, ***c*** and ***d*** are **real constants** and that ***x*** is the **independent variable**.

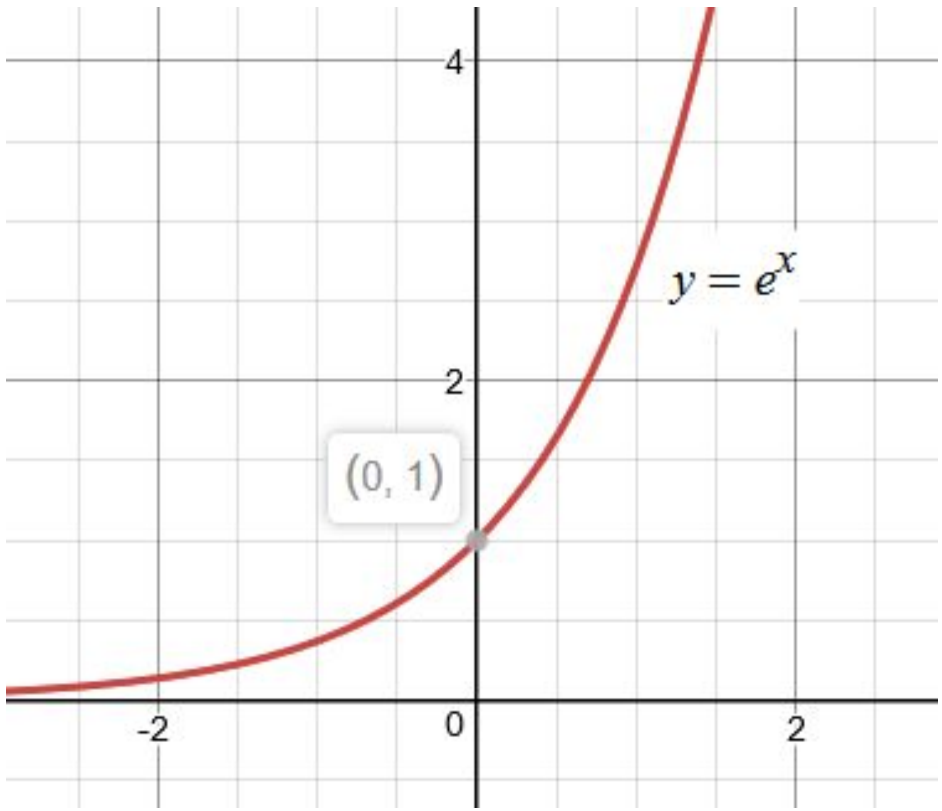
Let us see some examples:



Let us see some examples:

Are there any Asymptotes?

$y=e^x$

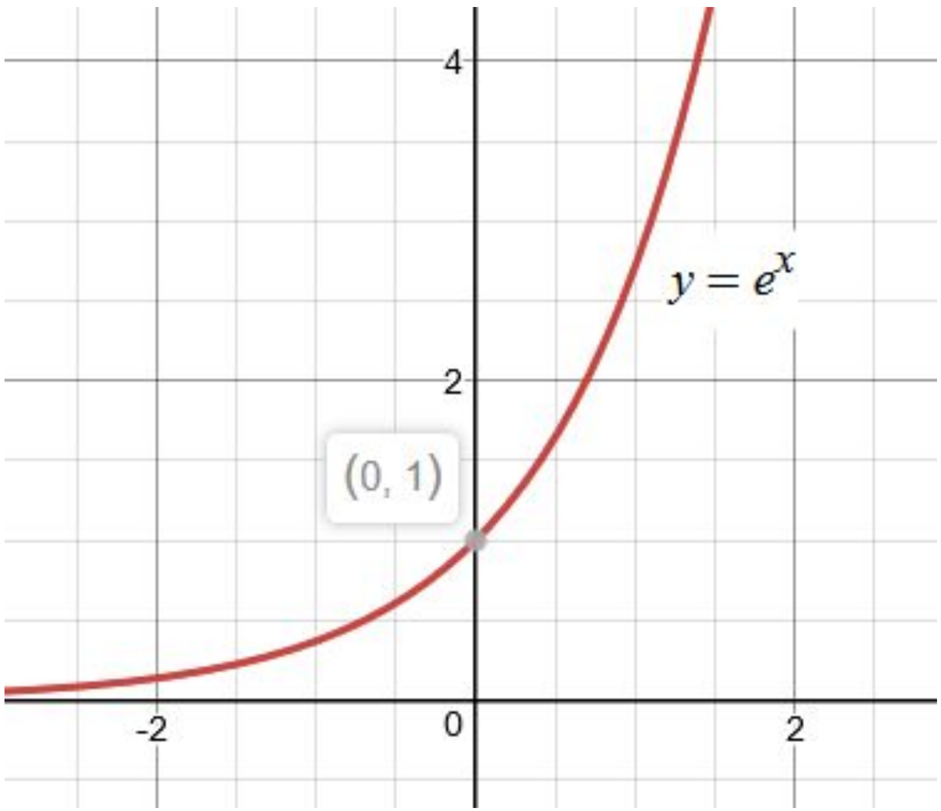


Let us see some examples:

Are there any **Asymptotes?** **HA: y=0**

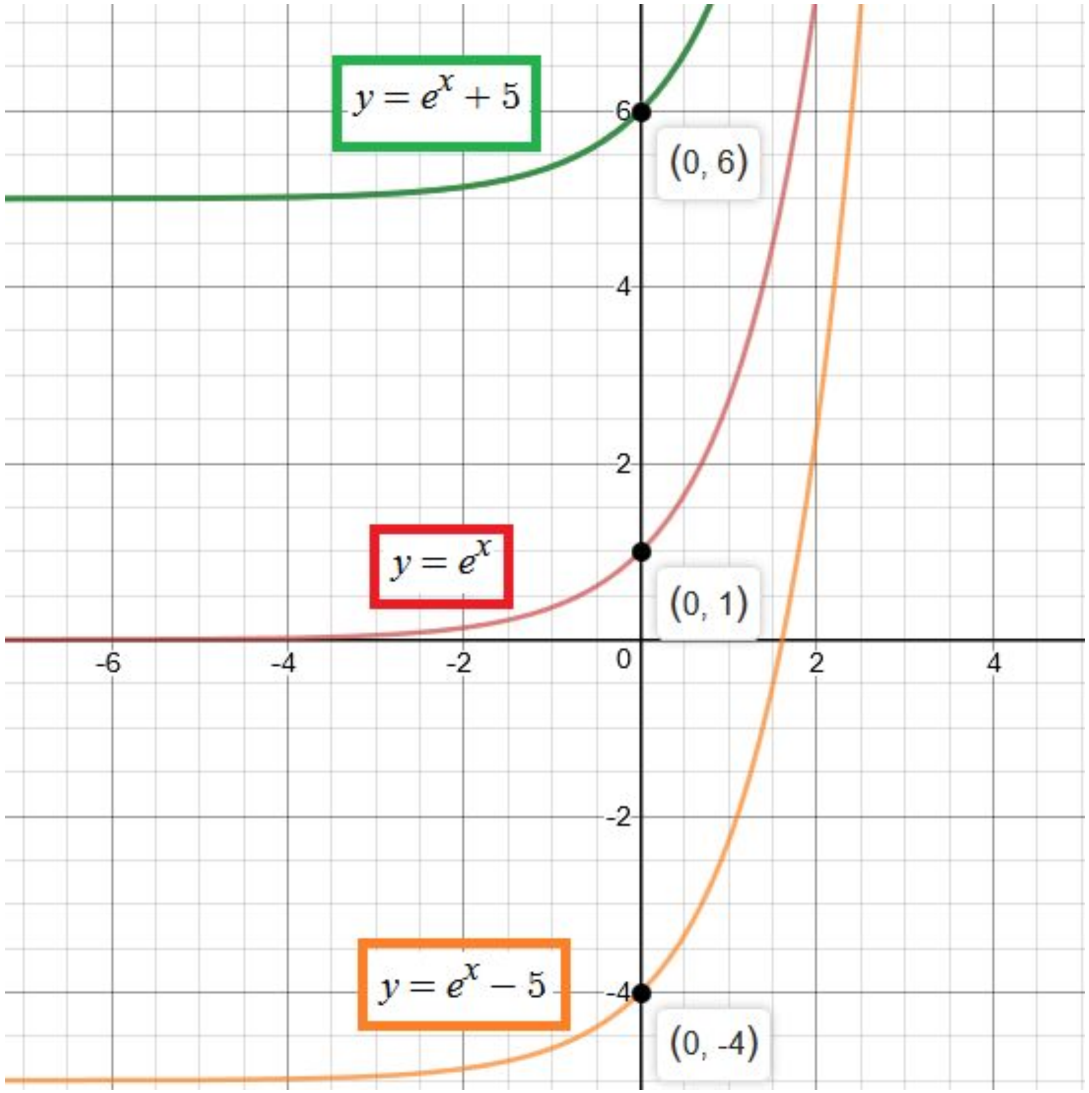
as $x \to -\infty$

$y=e^x$

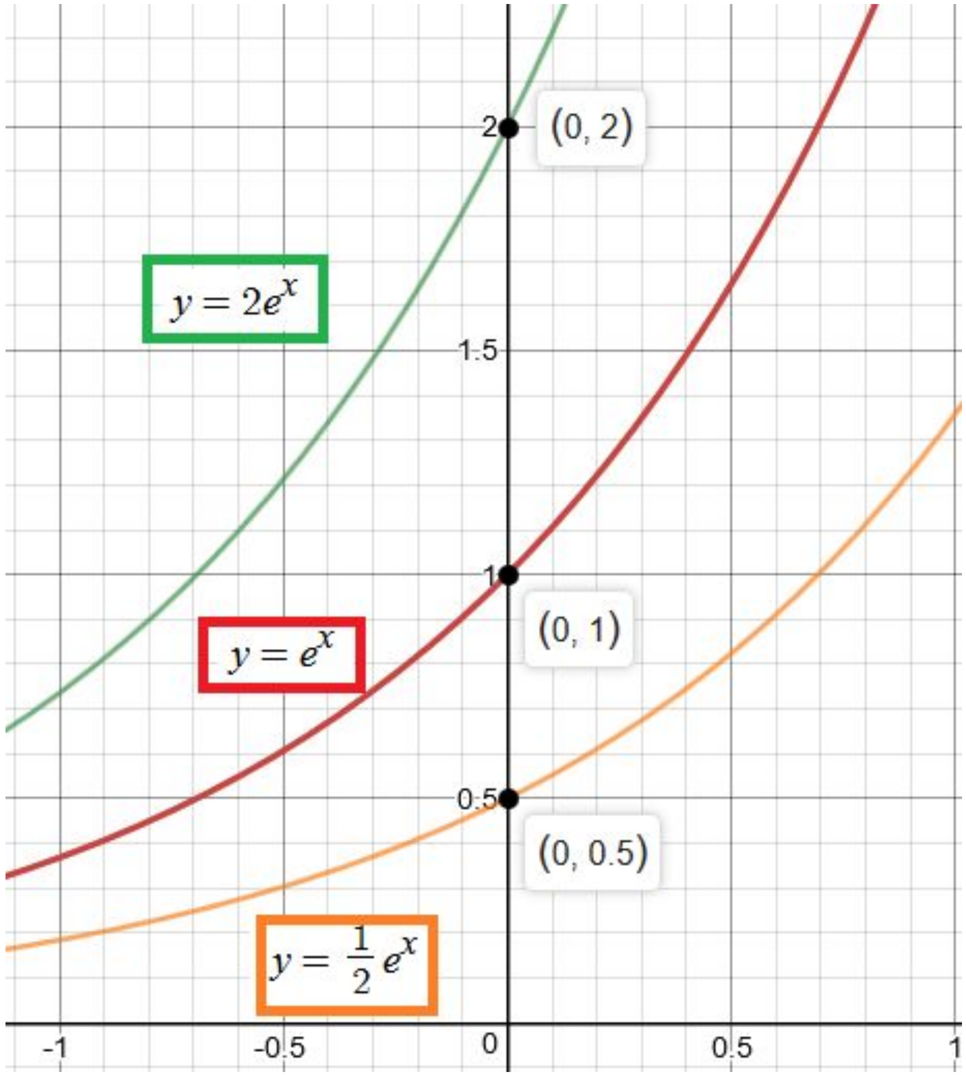


Note: HA (Horizontal asymptote)

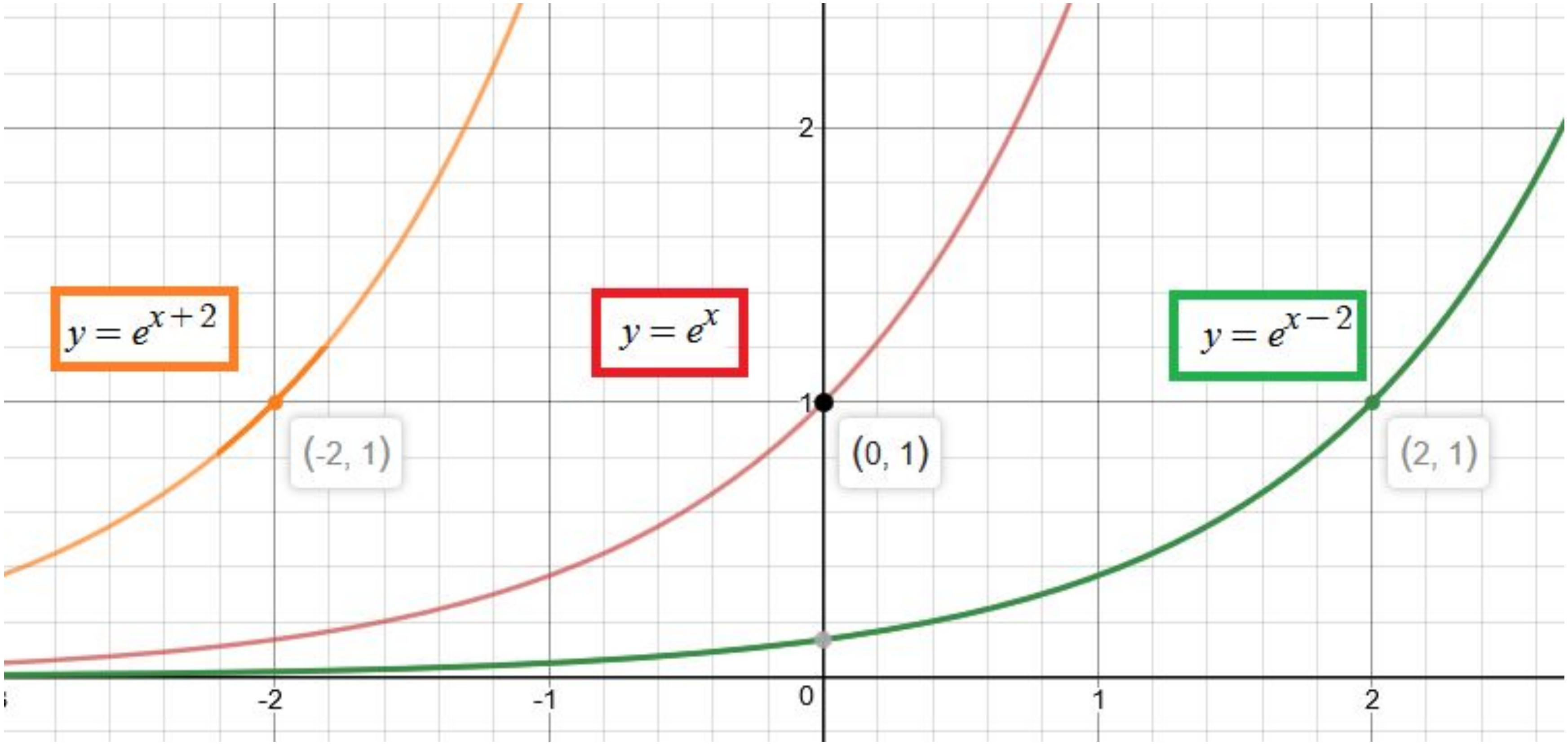
Vertical translation



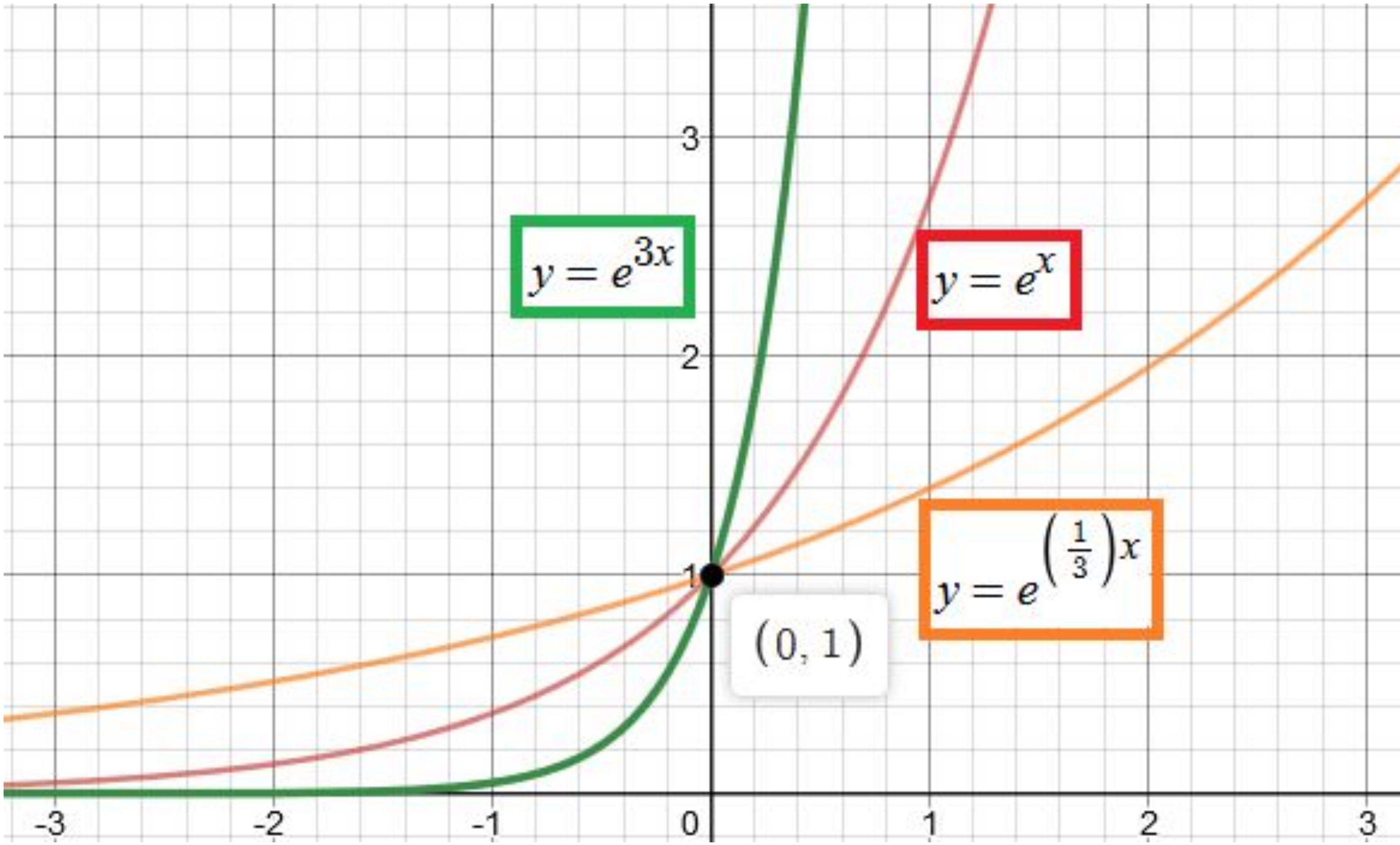
Vertical stretch



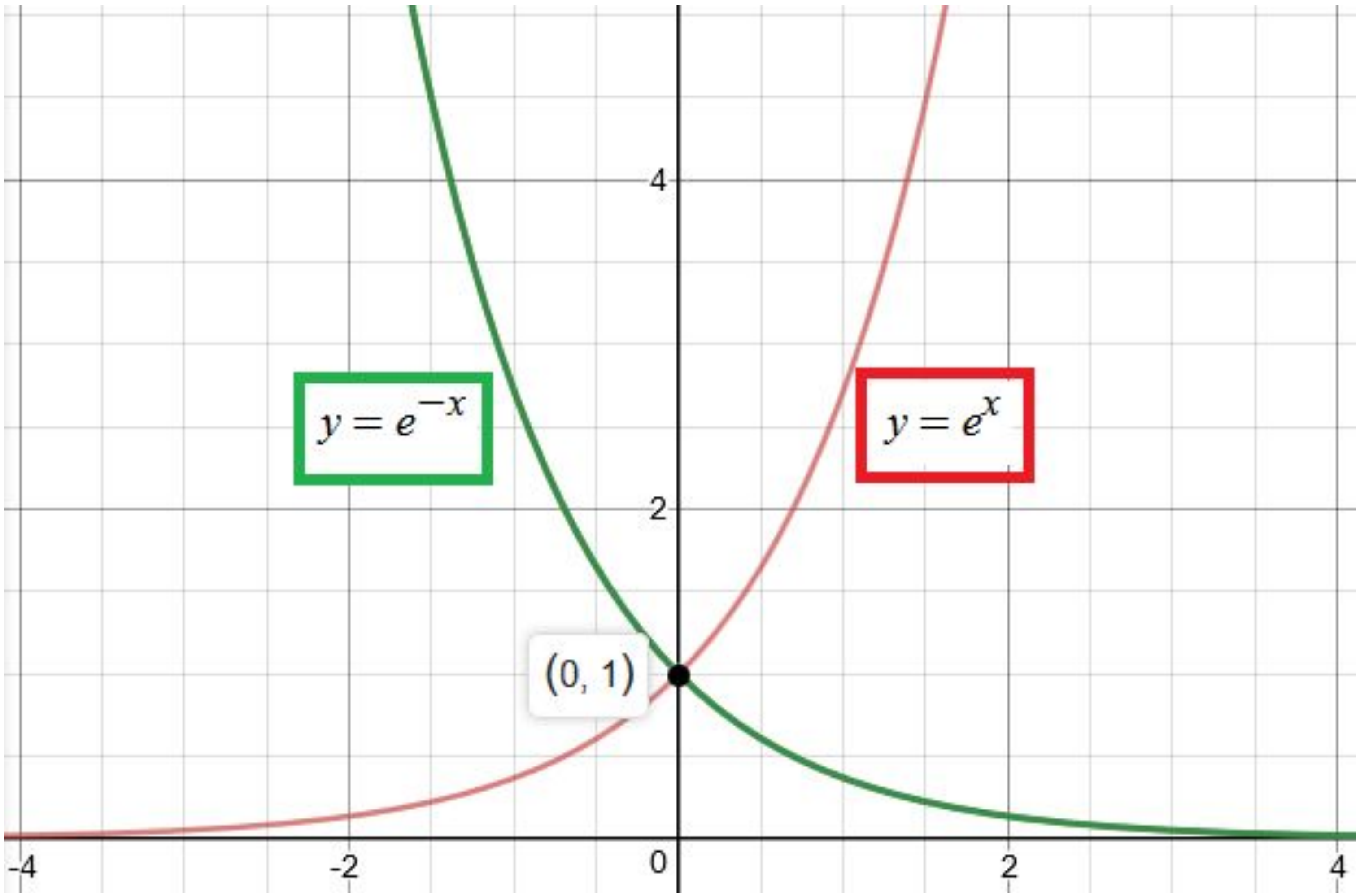
Horizontal translation



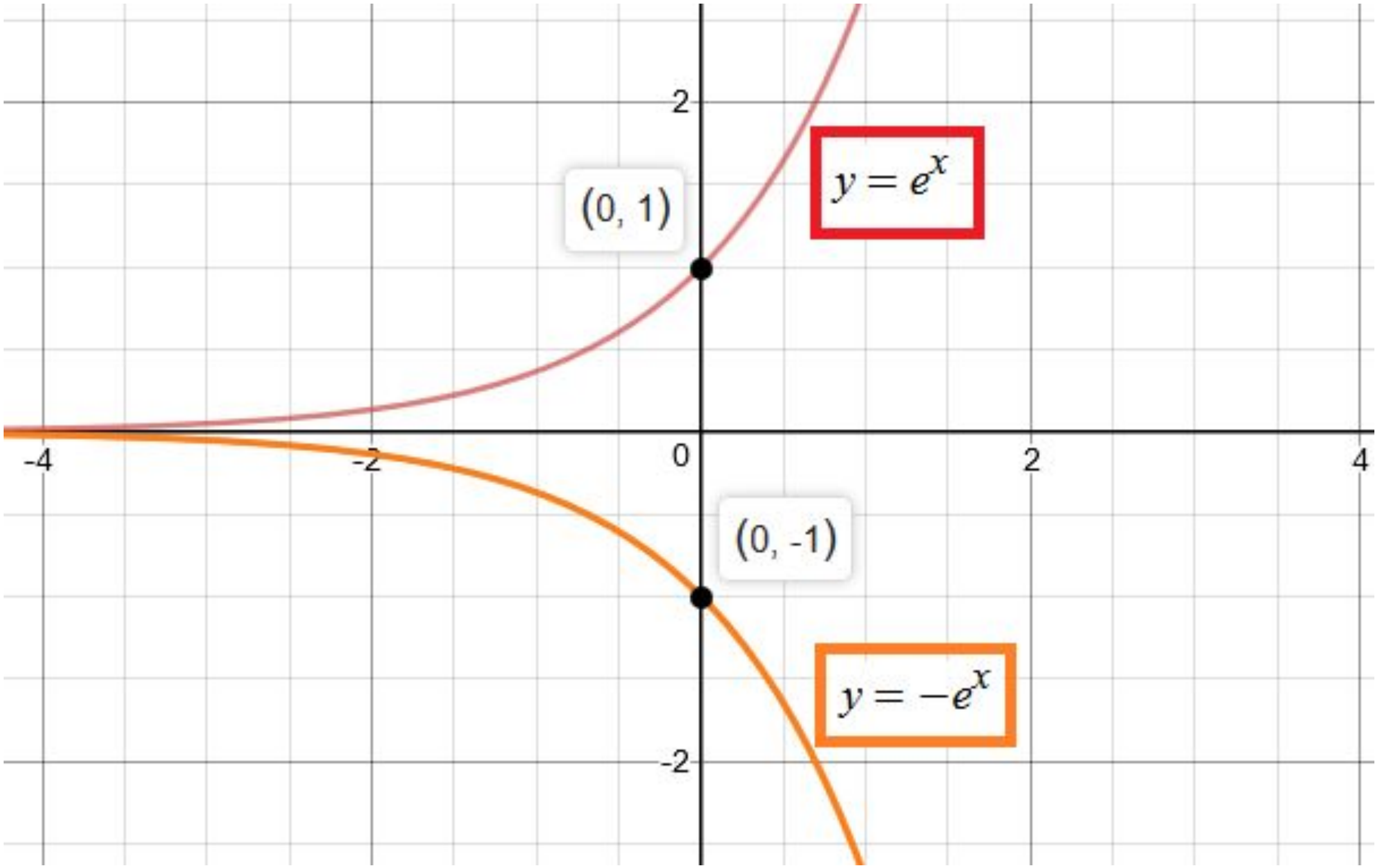
Horizontal stretch



Reflection in the y-axis (Horizontal)

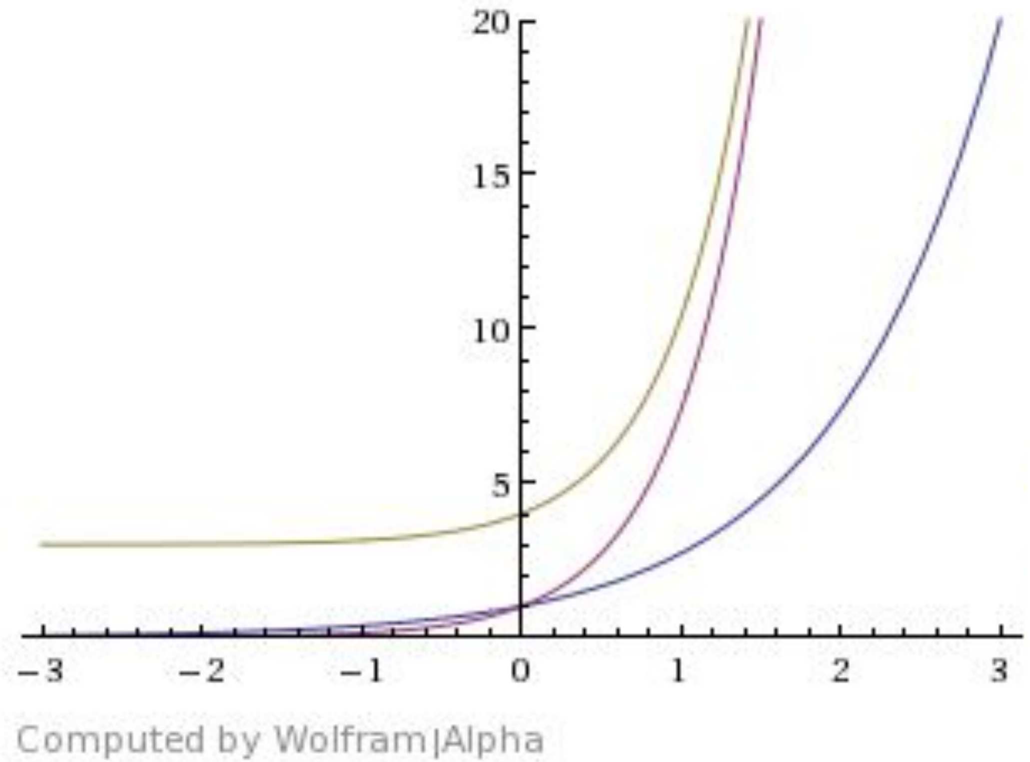


Reflection in the x-axis (Vertical)



Your turn!
Match function with its graph

1. $y=e^x$
2. $y=e^{2x}$
3. $y=3+e^{2x}$



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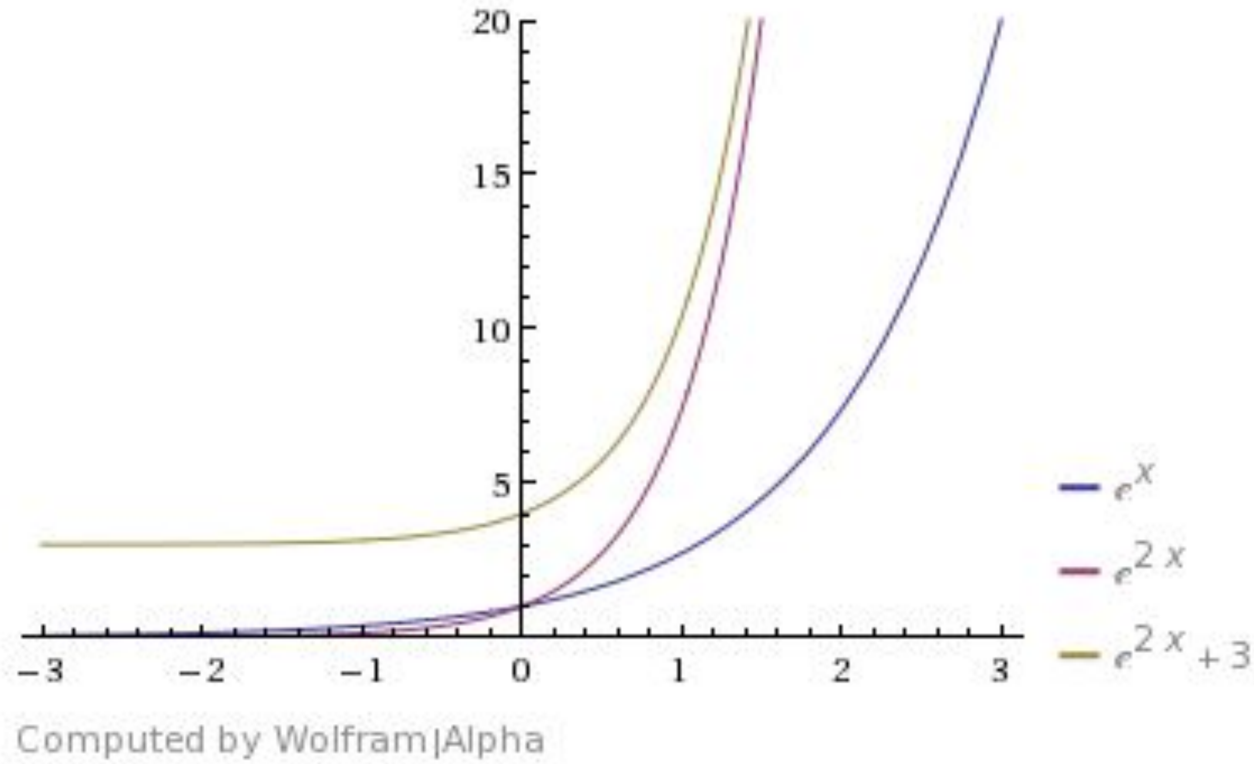
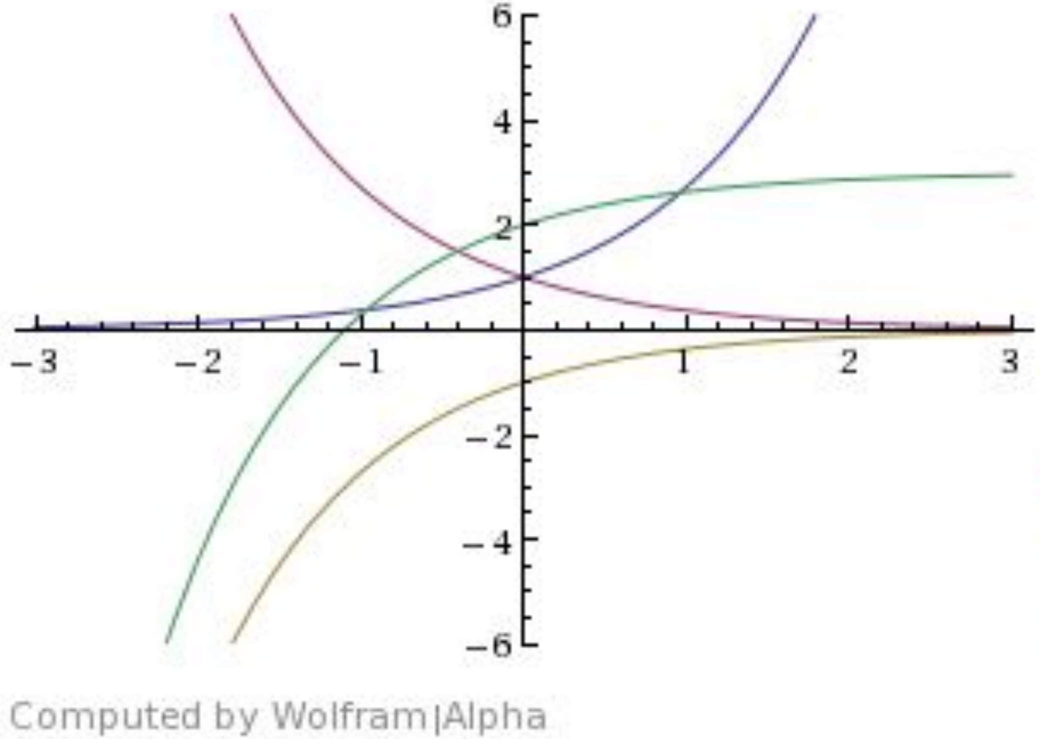


Your turn!
Match function with its graph

1. $y=e^{x}$
2. $y=e^{-x}$
3. $y=-e^{-x}$
4. $y=3-e^{-x}$



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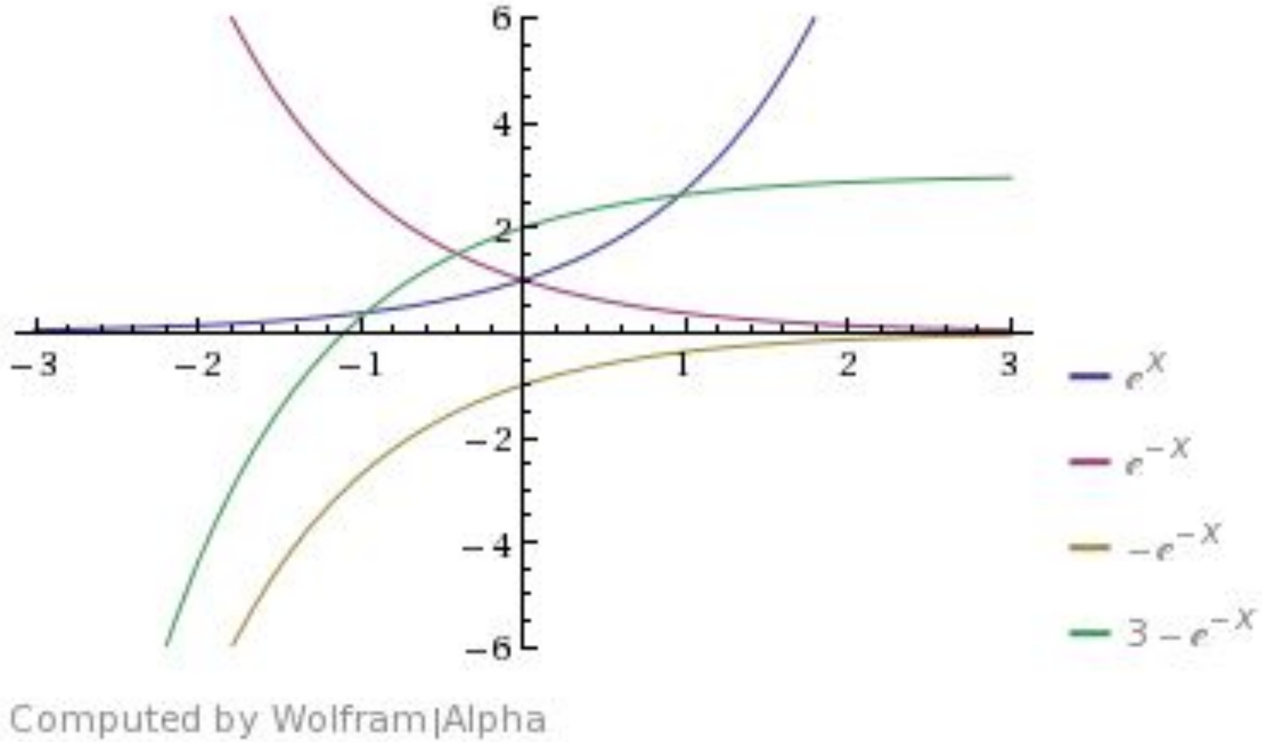
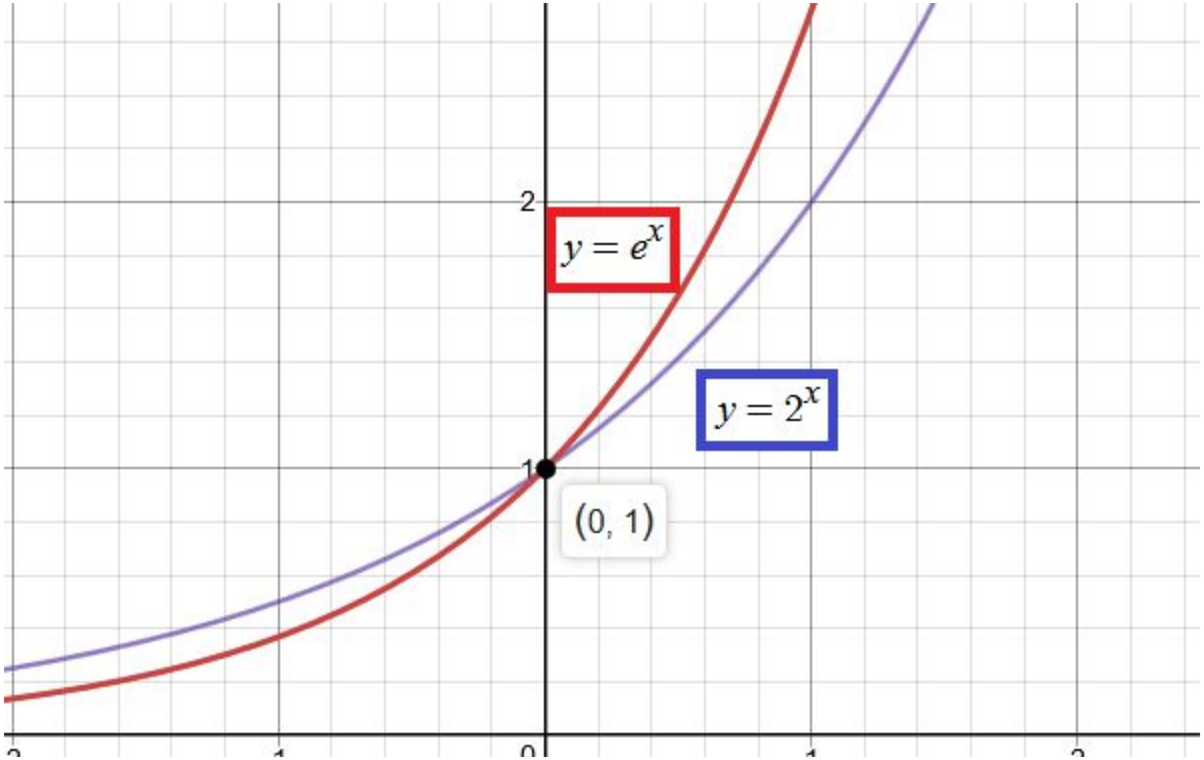


Have you noticed that we are now dealing with only base “e”?

$$y = a + be^{cx}$$

What is the reason for us to use only base “e”?

1. How to relate 2^x to e^x



2. What kind of transformation should be applied to e^x ?

Answer: we need to apply **horizontal stretch**, i.e. and introduce a coefficient **c**

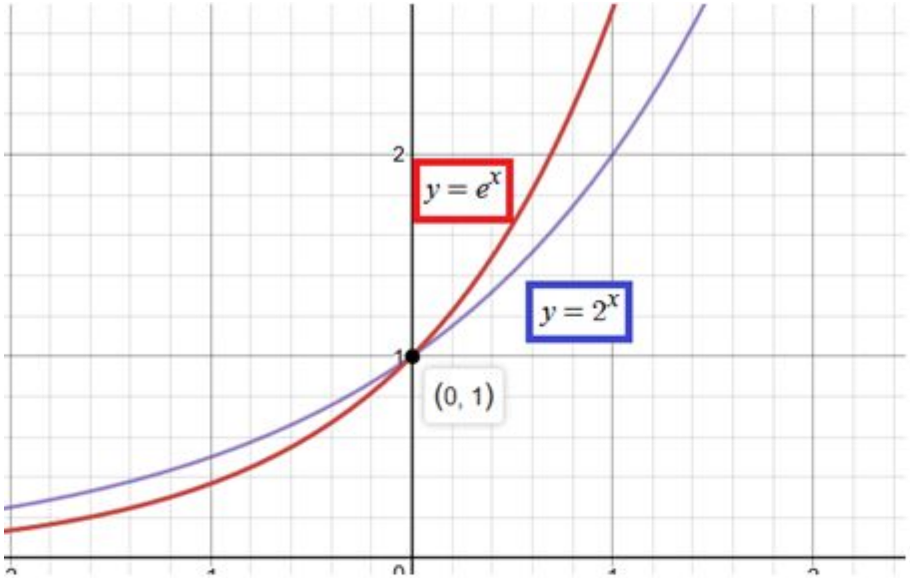
$$2^x = e^{cx}$$

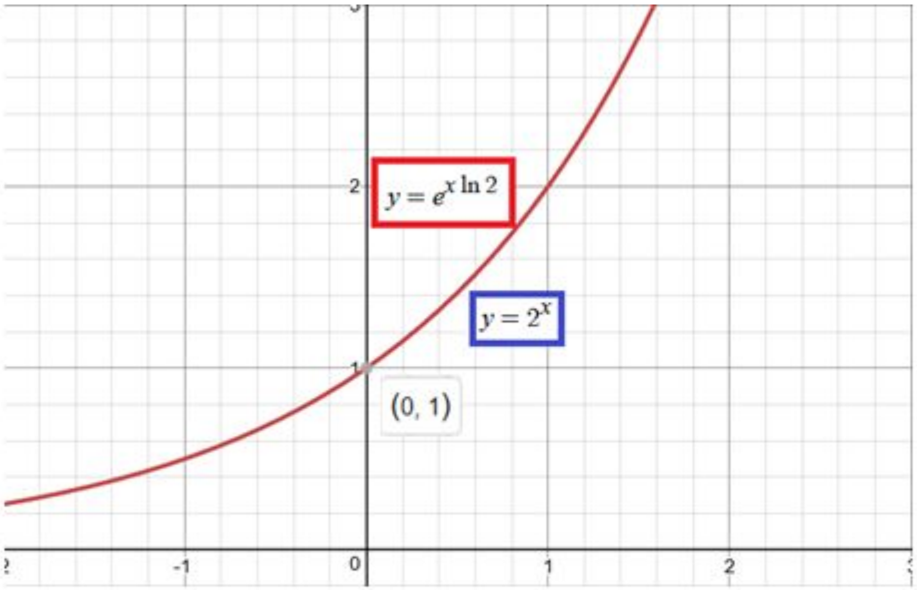
$$2 = e^c$$

$$\ln 2 = \ln e^c$$

$$c = \ln 2 = 0.693\ldots$$

$$2^x = e^{x \ln 2}$$





That is why in Exponential growth and decay models we use directly “e” number that can be tuned up to any numerical exponential number by horizontal stretch!

2.2.2 Sketch graphs of transformed natural logarithmic functions

Let's sketch graphs of transformed **exponential functions** such as:

$$y = a + b * ln(cx + d)$$

Horizontal scaling factor, scale factor 1/c

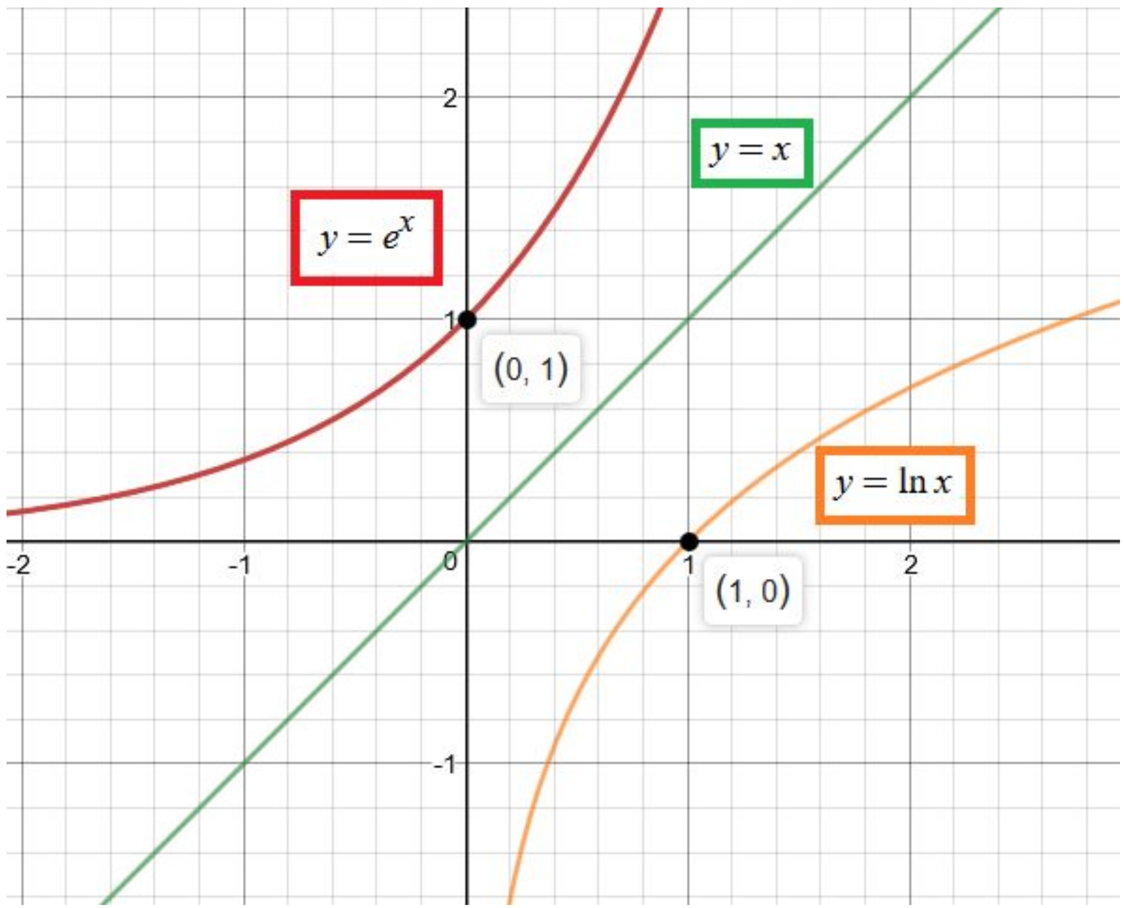
Vertical translation

Vertical scaling factor, scale factor b

Horizontal translation

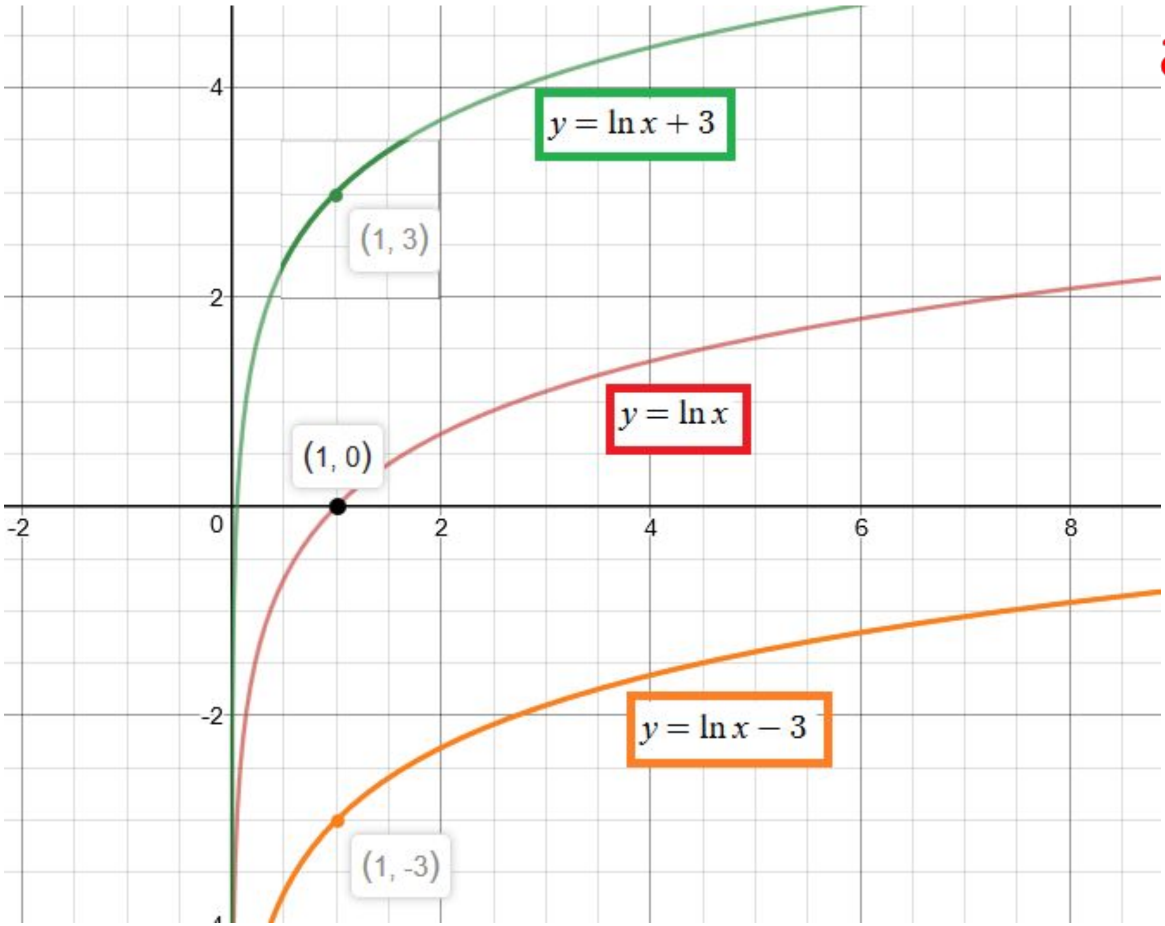
We assume that ***a***, ***b***, ***c*** and ***d*** are **real constants** and that ***x*** is the **independent variable**.

Let us see some examples:



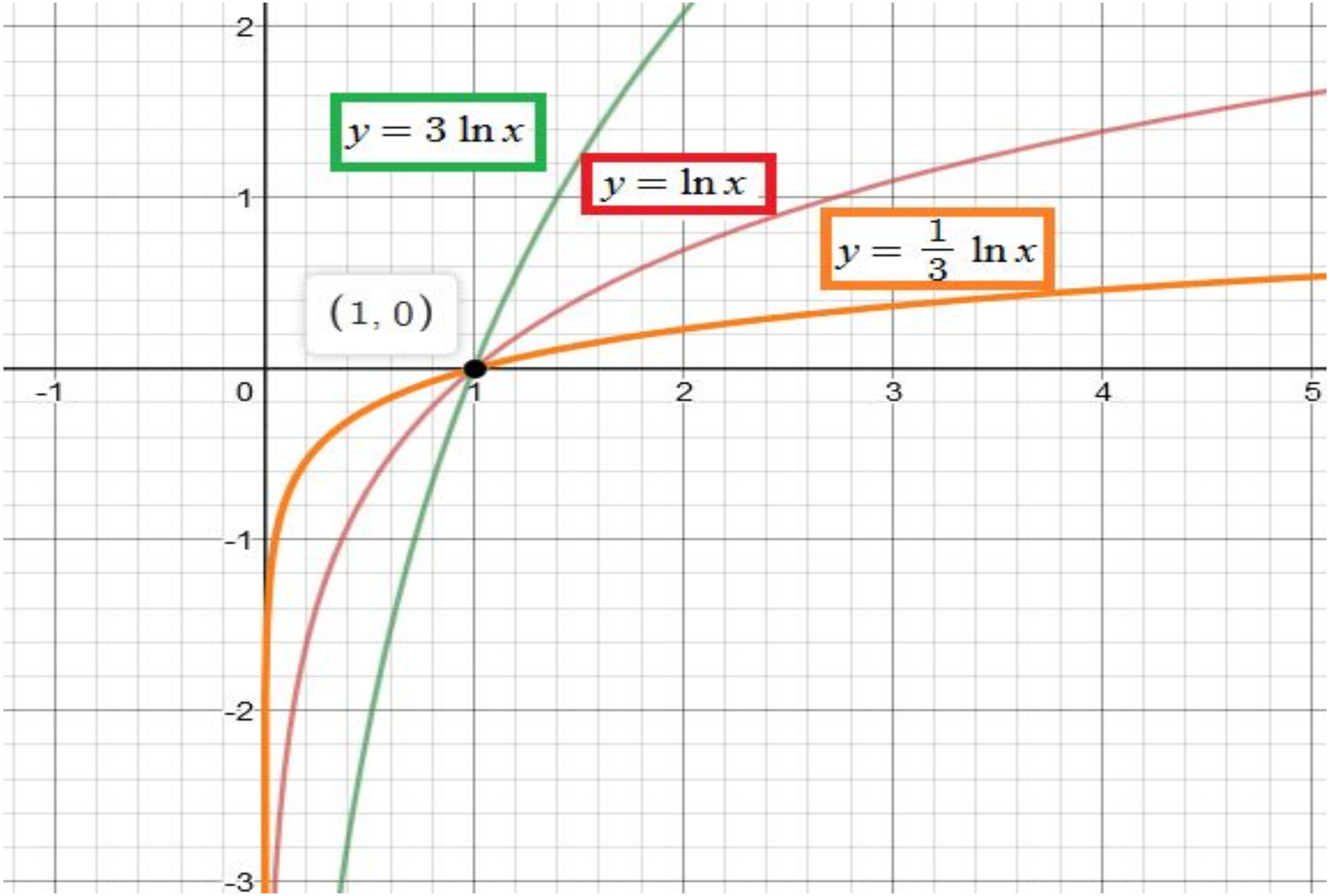
Vertical translation

Are there any **Asymptotes?** **VA: $x = 0$** as $x \to 0$



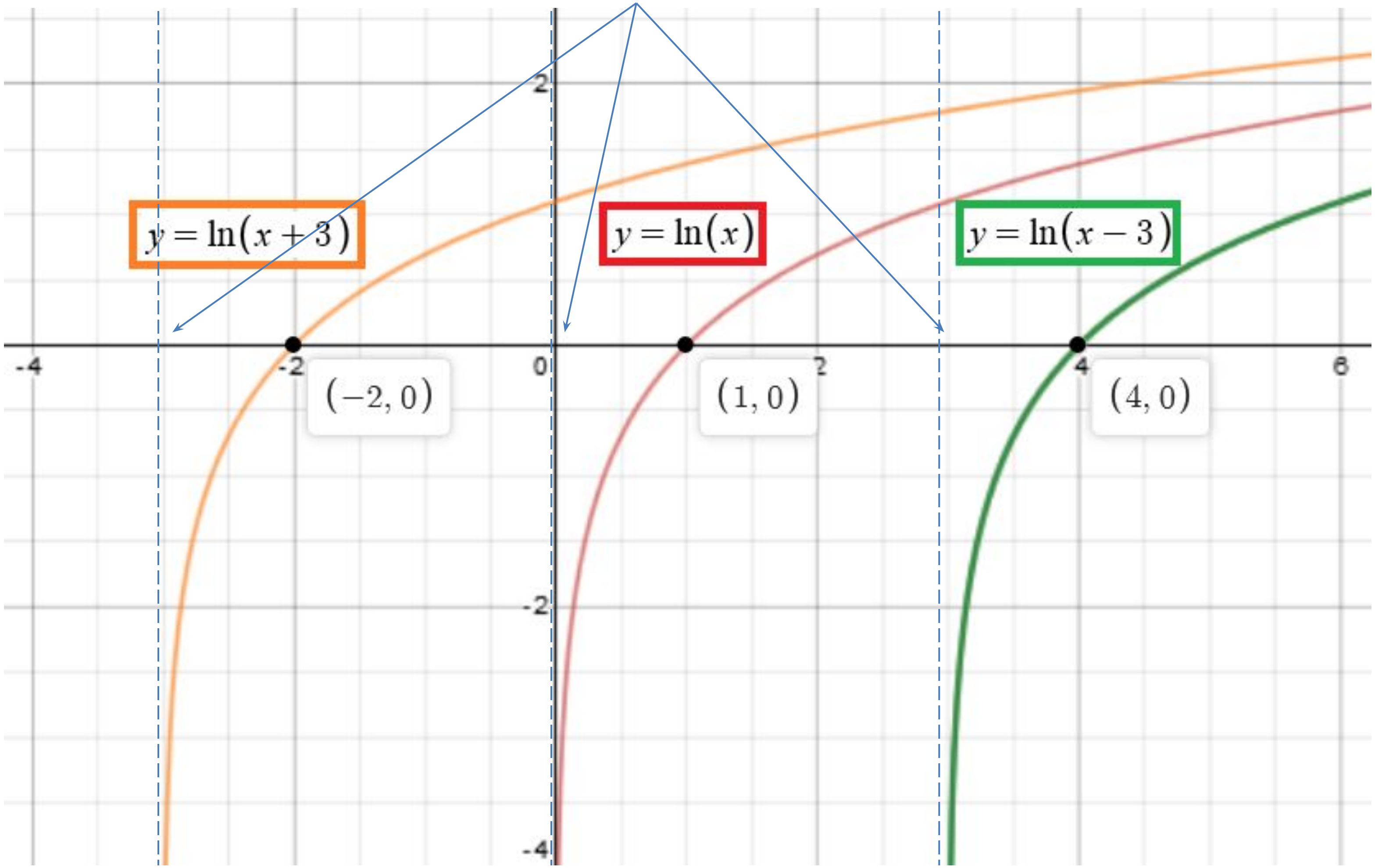
Note: VA (Vertical asymptote)

Vertical stretch

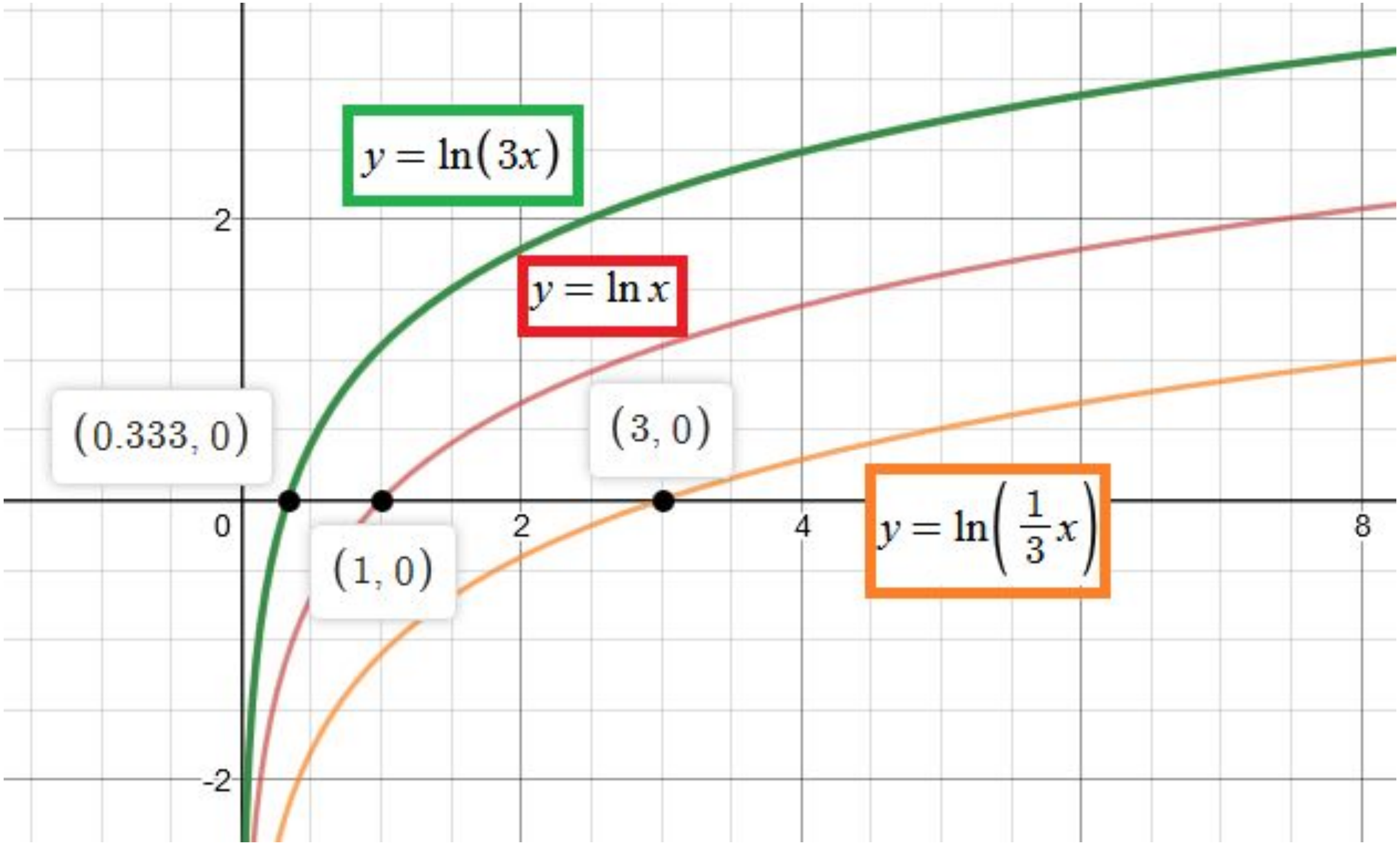


Horizontal translation

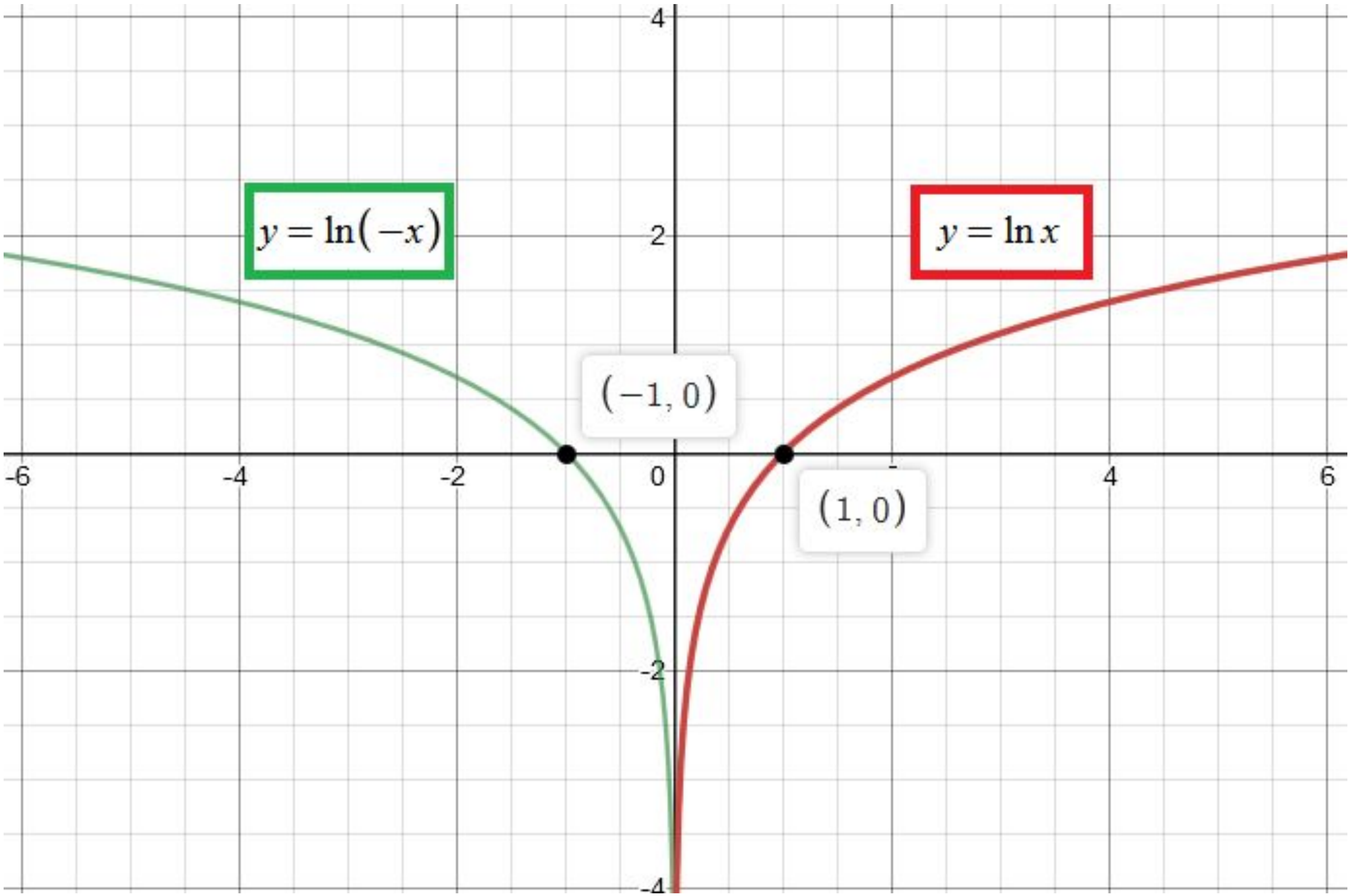
Asymptotes



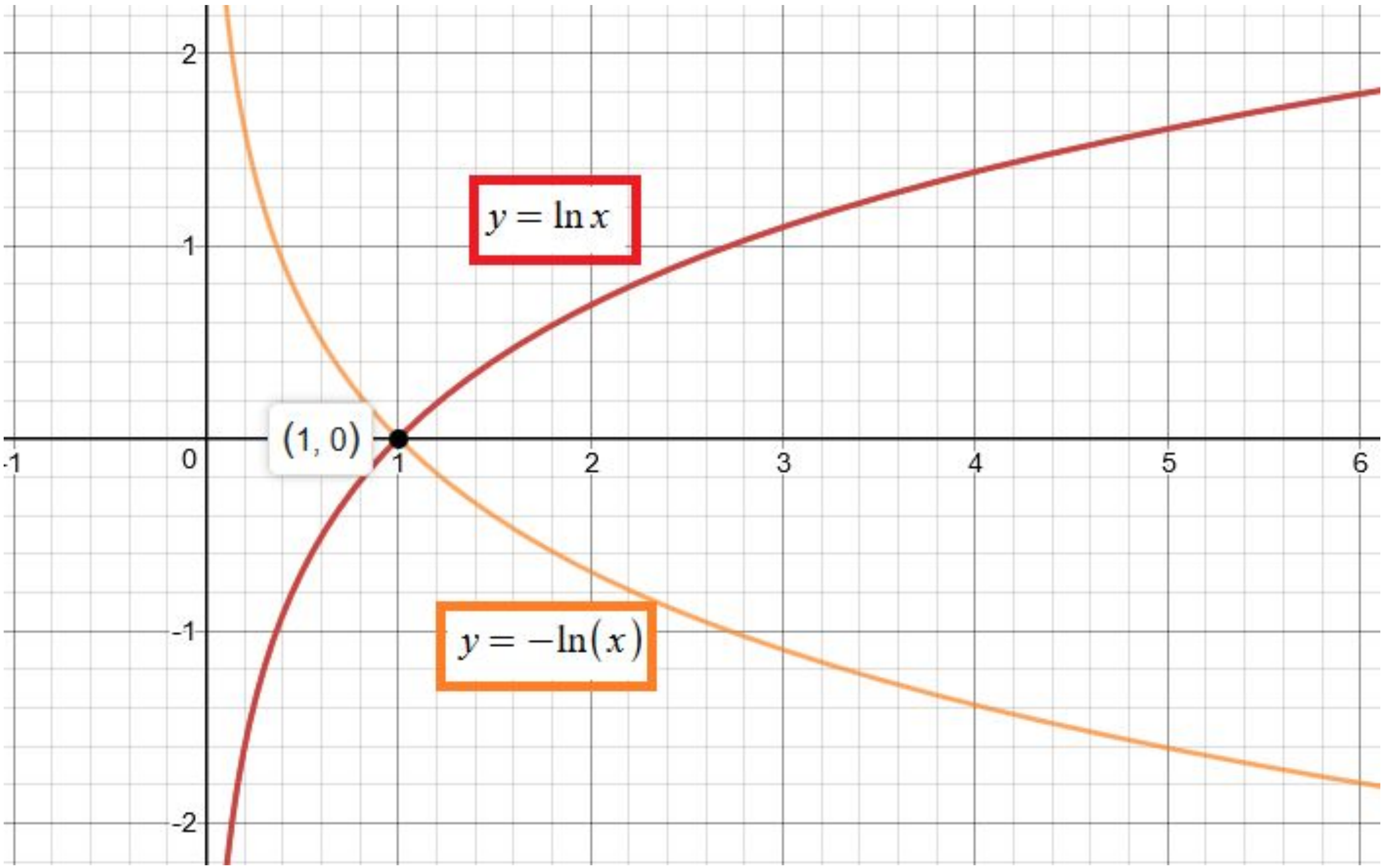
Horizontal stretch



Reflection in the y-axis (Horizontal)

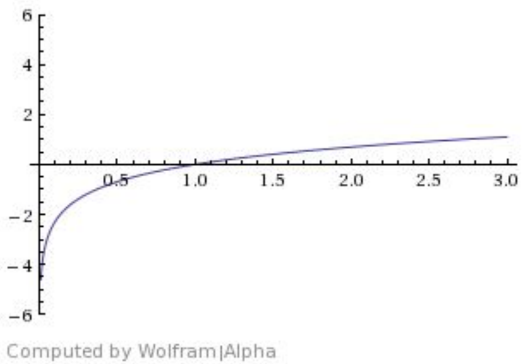


Reflection in the x-axis (Vertical)

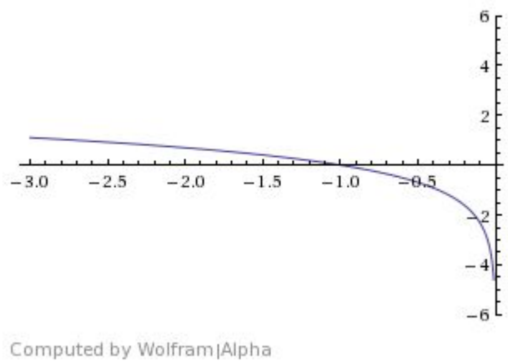


Let us see some examples:

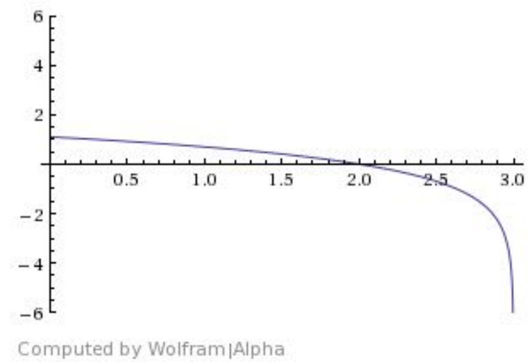
1. $y = \ln x$



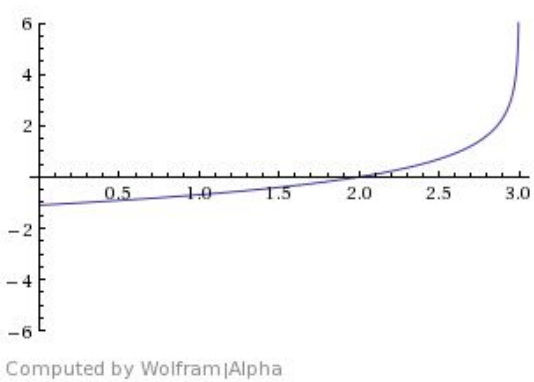
2. $y = \ln(-x)$



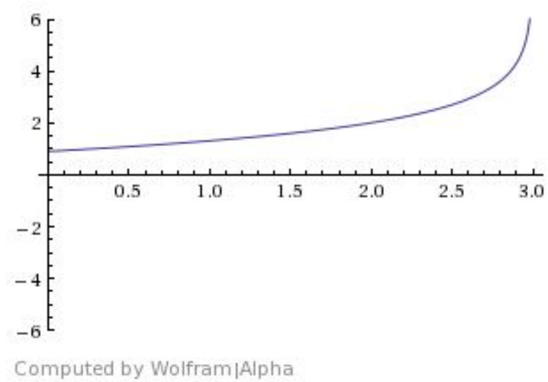
3. $y = \ln(3-x)$



4. $y = -\ln(3-x)$

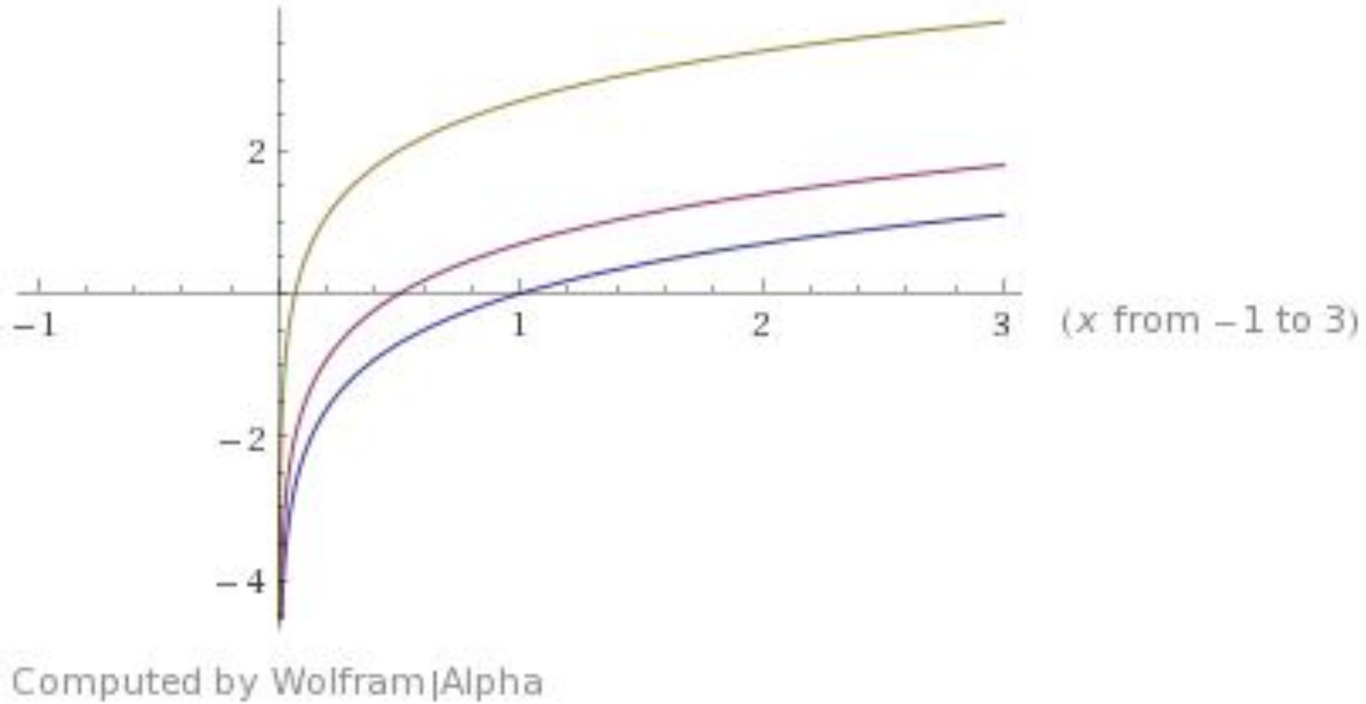


5. $y = 2-\ln(3-x)$



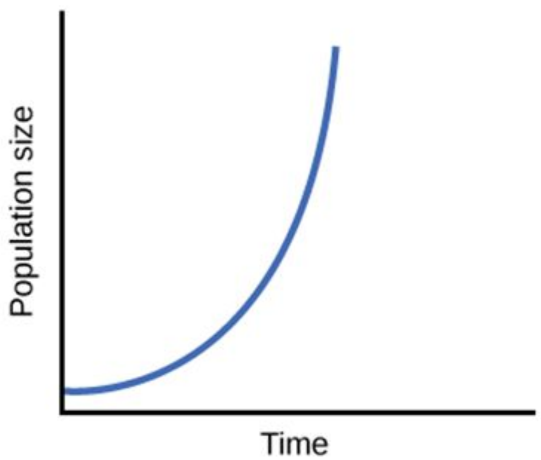
Your turn!
Match function with its graph

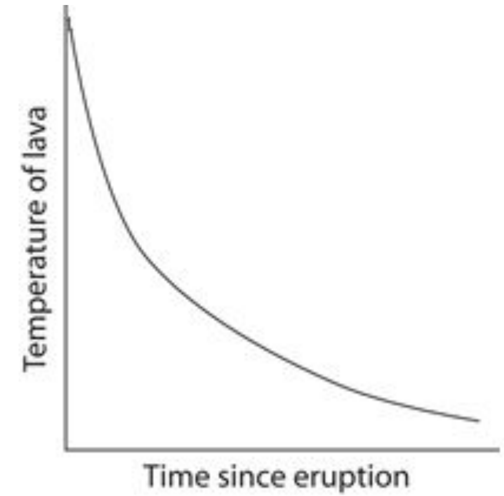
1. $y=\ln x$
2. $y=\ln(2x)$
3. $y=2+\ln(2x)$



2.2.3 Interpret and perform calculations with Exponential Growth and Decay models

Exponential Growth vs Exponential Decay





Note this difference

$$y = a * e^{bx}$$

$$y = a * e^{-bx}$$

Example 1 Decay model

The price of a used car can be represented by the formula

$$P = 16000e^{-\frac{t}{10}}$$

where P is the price in £'s and t is the age in years from new

Calculate:

a. The new price
b. The value at 5 years old car
c. What the model suggests about the eventual value of the car

Use this to sketch the graph of P against t.

Solutions:

a Substitute $t = 0$ into $P = 16\,000\, e^{-\frac{0}{10}}$

$= 16\,000 \times 1$

The new price is £16 000.

b Substitute $t = 5$ into $P = 16\,000\, e^{-\frac{5}{10}}$

$= 16\,000\, e^{-\frac{1}{2}}$

$= £9704.49$

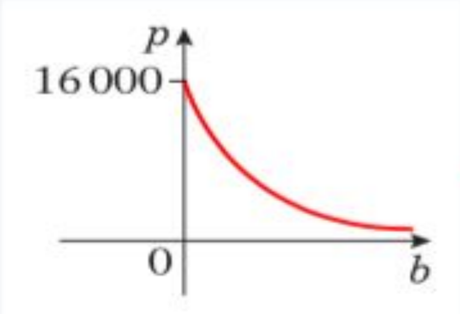
The price after 5 years is £9704.49.

c As $t \to \infty$, $e^{-\frac{t}{10}} \to 0$

Therefore $P \to 16\,000 \times 0 = 0$.

The eventual value is zero.

For the eventual value, let $t \to \infty$.



Use the values from parts **a**, **b** and **c** to sketch the graph.

Your turn (Example 2 Growth model)

The exponential growth of a colony of bacteria can modeled by he equation $A=60e^{(0.03t)}$ where, t is the time in hours from which the growth is recorded ($t \geq 0$)

a. Work out the initial population of bacteria.

b. Predict the number of bacteria after 4 hours.

c. Predict the time taken for the colony to grow to 1000.

Solutions:
$A=60e^{(0.03t)}$

a. Initial population

$A= 60e^{0.03(0)}=60e^{0}=60$ bacteria

b. When $t=4$

$A= 60e^{0.03(4)} \approx 60*1.1274\ldots$

$A \approx 68$ bacteria

c. After what time t will the number of bacteria be $A=1000$?

$1000=A= 60e^{0.03(t)}$

$e^{0.03(t)} = 16.67$

$\ln e^{0.03\,(t)} = \ln 16.67$

$0.03t = 2.8134$

$t = 93.8$ hours

Example 3 Decay model

Polonium-210 has a half-life of 140 days. Suppose a sample of this substances has a mass of 300mg.

a. Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t days.

b. Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t days.

c. Find the mass remaining after one year.

d. Draw a graph of the sample mass as a function of time.

Half–life is the time required for a quantity to reduce to half its initial value.

Solutions:

a. $m(t) = m_0 2^{-\frac{t}{h}}$

$$\frac{m_0}{2} = m_0 2^{-\frac{140}{h}}$$

$$\frac{1}{2} = 2^{-\frac{140}{h}}$$

$$\log_2 \frac{1}{2} = \log_2 2^{-\frac{140}{h}}$$

$$-1 = -\frac{140}{h}$$

$$h = 140$$

$$m(t) = 300 * 2^{-\frac{t}{140}}$$

b. $2^{-\frac{t}{140}} = e^{-rt}$

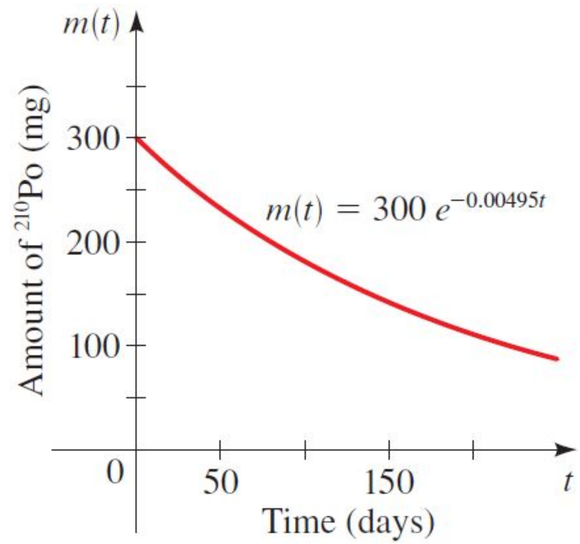
$$\ln 2^{-\frac{t}{140}} = \ln e^{-rt}$$

$$-r = -\frac{1}{140} \ln 2$$

$$r \approx 0.00495 \qquad m(t) = 300e^{-\frac{t}{140}\ln 2}$$

c. $m(365) = 300 * 2^{-\frac{365}{140}} \approx 49.24$

d.



Your turn (Example 4 Decay model)

The concentration, C of a drug in the blood stream, t hours after taking an initial dose, decreases exponentially according to $C = A(e^{-kt})$, where A and k are constants. If the initial concentration is 0.72 and this halves after 5 hours, find the values of A and k and sketch a graph of C against t.

Solutions:

When t=0; C=0.72

$0.72 = A\left(e^{-k*0}\right)$

$0.72 = A$

When t=5; C=0.36

$0.36 = 0.72\left(e^{-k*5}\right)$

$e^{5k} = \frac{0.72}{0.36} = 2$

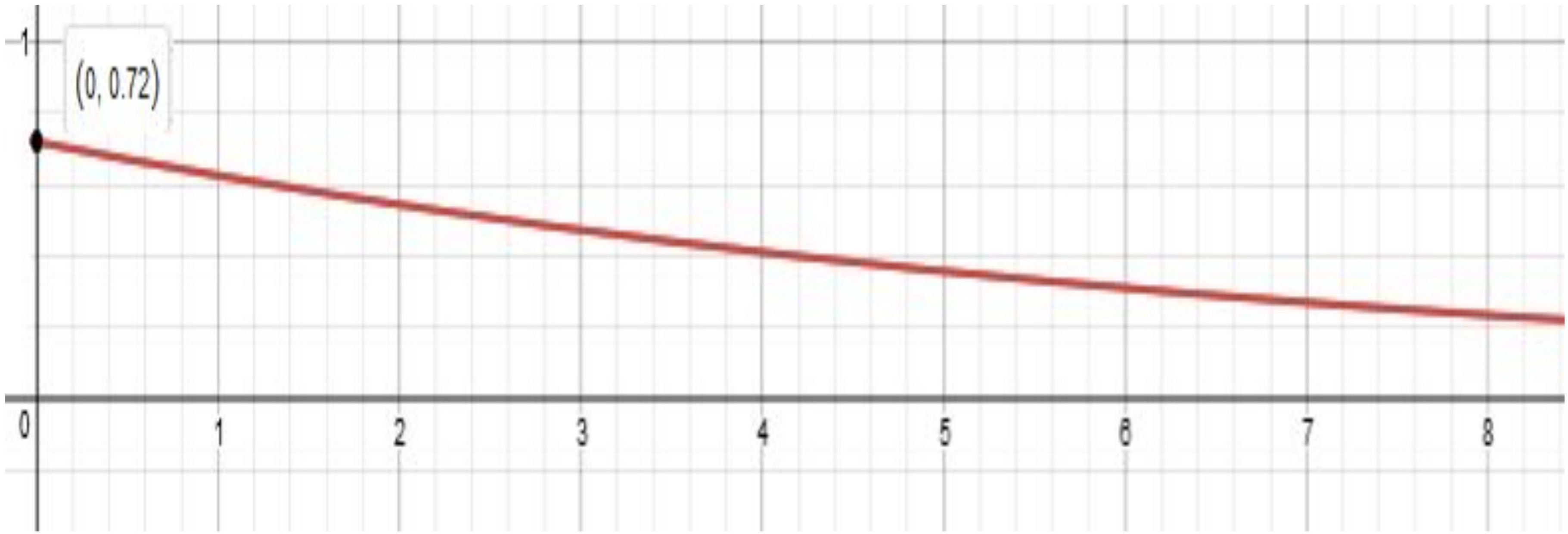
$ln\, e^{5k} = ln2$

$k = \frac{ln2}{5} \approx 0.14$

$C = 0.72(e^{-0.14*t})$

Solutions:

$$C = 0.72(e^{-0.14*t})$$



2.2.4 Solve applications involving Exponential and Logarithmic functions

Example 5 (Law of forgetting)

If a task is learned at a performance level P_0, then after a time interval t (in months) the performance level P staisfies

$$\log P = \log P_0 - c\log(t+1)$$

where c is a constant that depends on the type of task.

If your score on a mathematics test is 90, what score would you expect to get on a similar test (Assume that c is 0.2):

a. After two months?

b. After a year?

Solutions:

$$\log P = \log P_0 - c\log(t+1)$$
$$\log P = \log P_0 - \log(t+1)^c$$
$$\log P = \log \frac{P_0}{(t+1)^c}$$
$$P = \frac{P_0}{(t+1)^c}$$

a. After two months.

$$P(2) = \frac{90}{(2+1)^{0.2}} \approx 72$$

b. After a year.

$$P(12) = \frac{90}{(12+1)^{0.2}} \approx 54$$

Example 6 (Magnitude of earthquake)

The magnitude of the earthquake can be measured in Richter scale using: $M = \log \frac{I}{S}$

where I is the intensity of the earthquake and S is the intensity of "standard" earthquake.

The 1906 earthquake in San Francisco had an estimated magnitude of 8.3 on the Richter scale. In the same year a powerful earthquake occurred on the Columbia-Ecuador border that was four times as intense. What was the magnitude of the Columbia-Ecuador earthquake on the Richter scale?

Solutions:

If I_s is the intensity of the San Francisco earthquake, then from the definition of the magnitude we have

$$M = log\frac{I_s}{S} = 8.3$$

Intensity of Columbia-Ecuador earthquake $I_{ce} = 4I_s$

$$M = log\frac{I_{ce}}{S} = log\frac{4I_s}{S} = log\,4 + log\frac{I_s}{S} = log\,4 + 8.3 \approx 8.9$$

Your turn (Example 7)

The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. How many times more intense (in other words find $\frac{I_S}{I_L}$) was the San Francisco earthquake in 1906 than the Loma Prieta's earthquake in 1989?

Solutions:

I_s is the magnitude of the San Francisco earthquake.
I_L is the intensity of the Loma Prieta earthquake.

According to the question we are required to find $\frac{I_s}{I_L}$

$$\frac{I_s}{I_L} = \frac{\frac{I_s}{S}}{\frac{I_L}{S}} = 10^{\log\frac{\frac{I_s}{S}}{\frac{I_L}{S}}} = 10^{\log\frac{I_s}{S} - \log\frac{I_L}{S}} = 10^{M_s - M_L} = 10^{8.3-7.1} = 10^{1.2}$$

$$\approx 16 \text{ times}$$

Learning outcomes

At the end of this lecture, you should be able to:

2.2.1 **Sketch** the **graphs** of **transformed Exponential functions**

2.2.2 **Sketch** the **graphs** of **transformed Logarithmic functions**

2.2.3 **Interpret** and **perform** calculations with Exponential **Growth** and **Decay** models

2.2.4 **Solve applications** involving **Exponential** and **Logarithmic functions**

Preview activity 1: Trigonometry

Watch this video

https://www.youtube.com/watch?v=T9lt6MZKLck

Preview activity 2: Trigonometry

A measuring wheel with a radius of 25cm is used to measure a 30m distance. Calculate the angle in Rad, and find the number of full rotation it has to do?

$$l = r\theta$$