TECHNION
Israel Institute of Technology
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Resonator modes

CLASS EXERCISE 8

Long. And trans. Resonance frequencies

- Resonance frequency of the system:
  - Beam full round-trip ↔ phase *2πq* (where q is an integer)
- In the FP case this leads to:

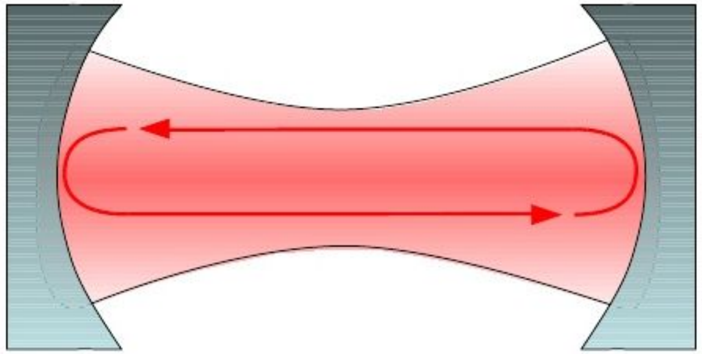
$$\nu_m = \frac{c}{2nL}$$

Long. And trans. Resonance frequencies

- In FP the mirrors are flat plane waves
- For curved mirrors the beams have transversal profile
- How does it change the solutions?

Long. And trans. Resonance frequencies

- Reminder: beams and mirrors curvatures are matched

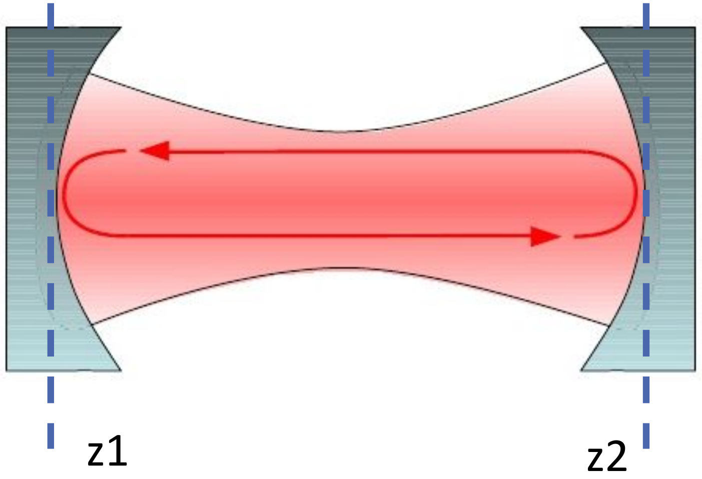


- This means that solving for *r=0* is enough

Long. And trans. Resonance frequencies

- The phase condition for half cycle is thus:

$$\Delta\phi = \theta_{qlm}(z_2) - \theta_{qlm}(z_1) = q\pi$$

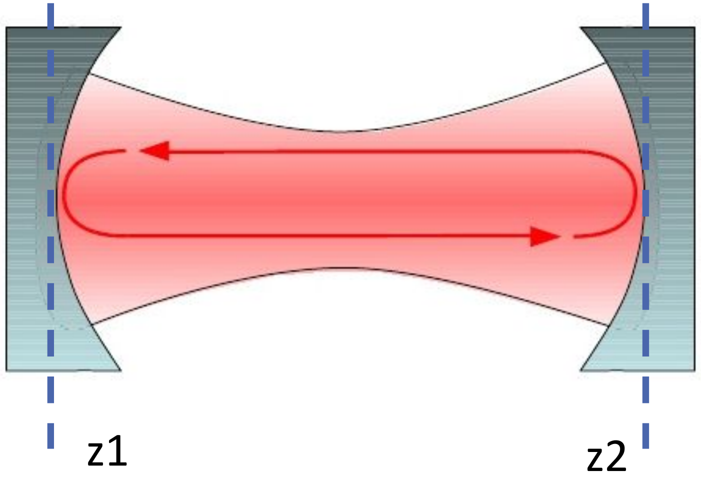


- The z-dependent phase of the beam is:

$$\theta_{qlm}(z) = k_{qlm}z - (l+m+1)\tan^{-1}\left(\frac{z}{z_0}\right)$$

Long. And trans. Resonance frequencies

- Thus we get:



$$\Delta\phi = k_{qlm}\left(z_2 - z_1\right) - \left(l + m + 1\right)\left(\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right)\right) = q\pi$$

$$\Rightarrow k_{qlm}L - \left(l + m + 1\right)\Delta\eta = q\pi$$

Long. And trans. Resonance frequencies

- From this equation we learn:

$$k_{qlm}L-\left(l+m+1\right)\Delta\eta=q\pi$$

  - The phase depends on *q*
  - The phase depends on transverse characteristics (*l*,*m*)

Long. And trans. Resonance frequencies

- We divide the solution into 2 cases: $$k_{qlm}L-\left(l+m+1\right)\Delta\eta=q\pi$$
  - Constant *l,m*
  - Constant *q*

Constant *l,m* – Longitudinal modes

- We write the equation for *q* and *q+1*: $k_{qlm}L-(l+m+1)\Delta\eta = q\pi$

$$\begin{cases} k_q L-(l+m+1)\Delta\eta = q\pi \\ k_{q+1}L-(l+m+1)\Delta\eta = (q+1)\pi \end{cases}$$

$$\Rightarrow (k_{q+1}-k_q)L = \pi$$

$$\Rightarrow \frac{2\pi nL}{c}(\nu_{q+1}-\nu_q) = \pi$$

$$\Rightarrow \nu_{q+1}-\nu_q = \frac{c}{2nL}$$

Constant *l,m* – Longitudinal modes

- We got:

$$\nu_{q+1} - \nu_q = \frac{c}{2nL} \equiv FSR_{FP}$$

- Which is exactly the FSR of a FP resonator
- These modes depend only on the length of the resonator
- □they are called, thus, Longitudinal modes

Constant *q* – Transverse modes

- We write the equation for 2 gaussian modes: $k_{qlm}L-(l+m+1)\Delta\eta = q\pi$

$$\begin{cases} k_{qlm}L-(l+m+1)\Delta\eta = q\pi \\ k_{ql'm'}L-(l'+m'+1)\Delta\eta = q\pi \end{cases}$$

$$\Rightarrow \left(k_{ql'm'}-k_{qlm}\right)L = \Delta(l+m)\Delta\eta$$

$$\Rightarrow \frac{2\pi nL}{c}\left(\nu_{ql'm'}-\nu_{qlm}\right) = \Delta(l+m)\Delta\eta$$

$$\Rightarrow \nu_{ql'm'}-\nu_{qlm} = \frac{c}{2nL}\frac{\Delta\eta}{\pi}\Delta(l+m)$$

Constant *q* – Transverse modes

- We got:

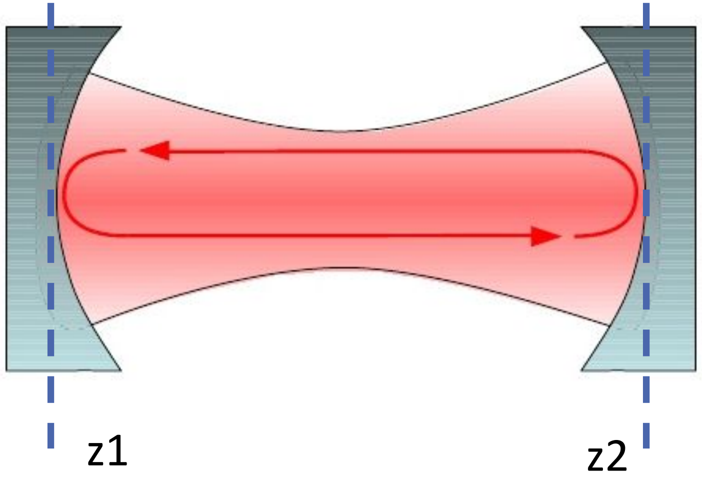
$$\nu_{ql'm'} - \nu_{qlm} = \frac{c}{2nL}\frac{\Delta\eta}{\pi}\Delta\left(l+m\right) = FSR_{FP}\frac{\Delta\eta}{\pi}\Delta\left(l+m\right)$$

- The result is invariant to switching *l* and *m*
- Depends on difference in transverse profile (subtraction of *l+m)*
- □they are called, thus, Transverse modes

Examples – symmetric resonator

- Symmetric resonator:

$$z_1 = -z_2$$



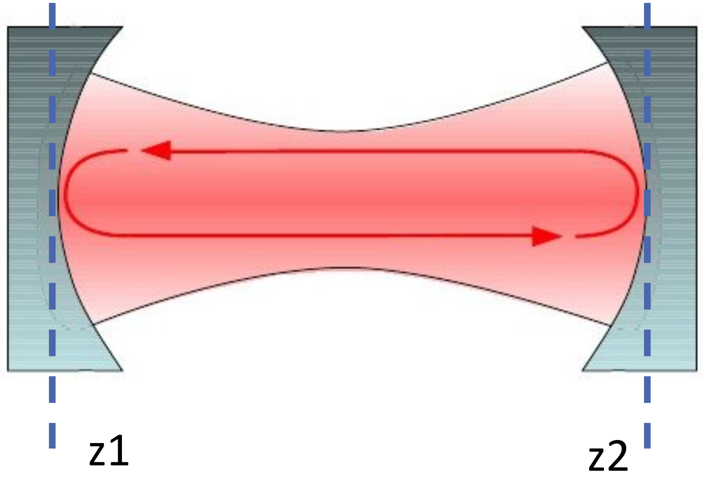
  - Thus we have:

$$\Delta\eta = \tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right) = 2\tan^{-1}\left(\frac{z_2}{z_0}\right)$$

Examples – confocal symmetric resonator

- Confocal symmetric resonator:

  - If the resonator is also confocal:



$$z_1 = -z_2 \Rightarrow z_2 = L/2$$
$$R = L$$

Examples – confocal symmetric resonator

- Solving L as a function of z_0:

$$z_1 = -z_2 \Rightarrow z_2 = L/2$$

$$R = L$$

$$R = R\left(z_2\right) = z_2 + \frac{{z_0}^2}{z_2} = L$$

$$\Rightarrow \frac{L}{2} + 2\frac{{z_0}^2}{L} = L$$

$$\Rightarrow {z_0}^2 = L^2/4$$

$$\Rightarrow z_0 = L/2$$

Examples – confocal symmetric resonator

- Since the resonator is symmetric:

$$\Delta\eta = 2\tan^{-1}\left(\frac{z_2}{z_0}\right)$$

$$= 2\tan^{-1}\left(\frac{L/2}{L/2}\right) = \frac{\pi}{2}$$

$$\Rightarrow \nu_{ql'm'} - \nu_{qlm} = \frac{c}{2nL}\frac{1}{2}\Delta(l+m) = FSR_{FP}\frac{1}{2}\Delta(l+m)$$

Examples – confocal symmetric resonator

$$\nu_{ql'm'} - \nu_{qlm} = \frac{c}{2nL}\frac{1}{2}\Delta(l+m) = FSR_{FP}\frac{1}{2}\Delta(l+m)$$

- Resonance frequencies can:
  - Coincide with original modes
  - Be between two modes
- The number of modes in a section is doubled

Examples – nearly planar resonator

- We assume:

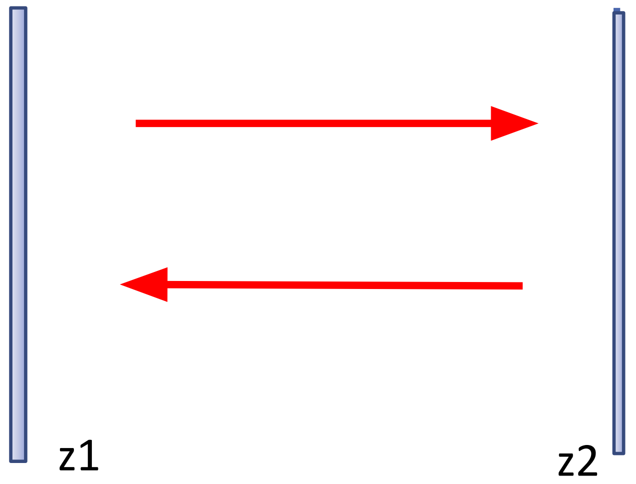
$$R \;\square\; L$$

- Thus we have:

$$R = R\left(z_2\right) = z_2 + \frac{z_0^{\;2}}{z_2} \;\square\; L$$

- This leads to either:

$$\begin{cases} z_2 \;\square\; L \\ z_2 \;\square\; z_0 \end{cases}$$



Examples – nearly planar resonator

- The first option is impossible since by definition $z_2 < L$
- Thus given $z_2 \ll z_0$ we have:

$$\Delta\eta = \tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right) \approx \frac{z_2}{z_0} - \frac{z_1}{z_0} = \frac{z_2 - z_1}{z_0}$$

$$\Rightarrow \Delta\eta = \frac{L}{z_0}$$

Examples – nearly planar resonator

- So the resonance frequencies are:

$$\nu_{ql'm'} - \nu_{qlm} = \frac{c}{2n\pi z_0}\Delta\left(l+m\right)$$

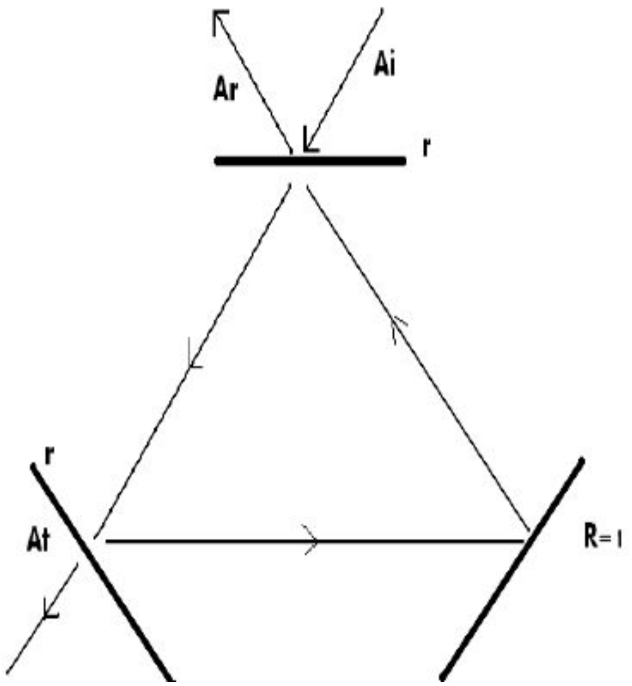
- Since $z_0 >> L$ we have many frequencies between long. freqs.

- This is undesirable since quality and coherence are determined

  by the number of operating modes

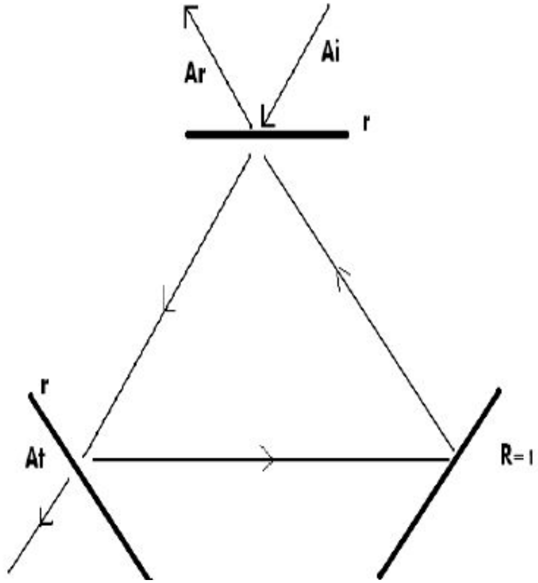
A circular resonator

- Given by 3 mirrors on the vertices of an equilateral triangle



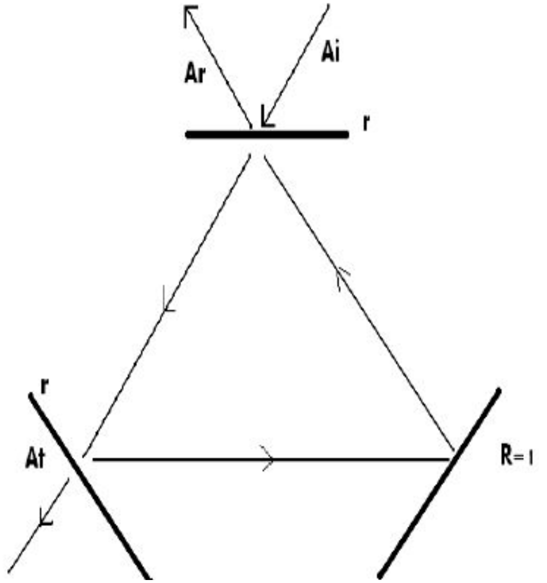
A circular resonator

- The upper (entrance) and left (exit) mirrors are dielectric mirrors with: $r=-r'$
- The right mirror is fully reflective with $R=1$
- Notice that reflections add π phase and the perimeter of the triangle is L



A circular resonator

- **What are the transmission intensity and the resonance frequencies?**



A circular resonator

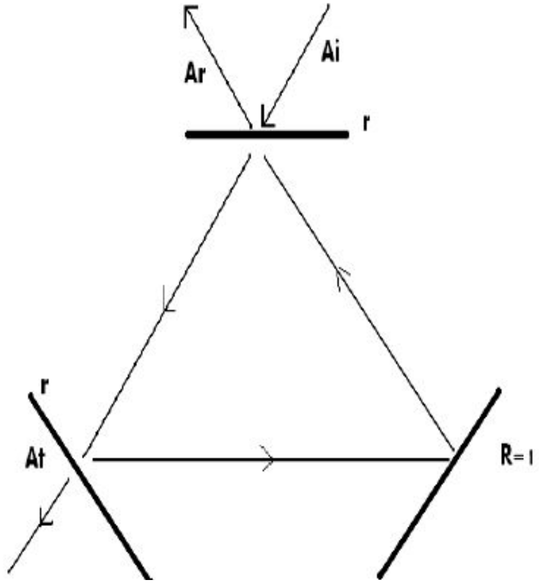
- We calculate the transmission by adding transmitted waves

  as we did for FP:

$$A_1 = A_i t e^{ikL/3} t'$$

$$A_2 = A_1 \left(-r'\right)^2 \left(-1\right) e^{ikL}$$

$$A_3 = A_2 \left(-r'\right)^2 \left(-1\right) e^{ikL}$$



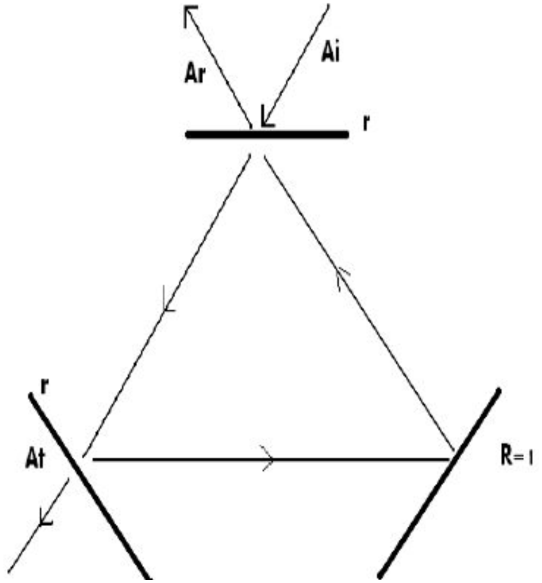
- And so on

A circular resonator

- Summing over all the partial waves:

$$A_t = \sum_j A_j = A_i T e^{ikL/3} \left[1 - r'^2 e^{ikL} + r'^4 e^{2ikL} - \ldots \right]$$

$$\frac{A_t}{A_i} = \frac{1-R}{1+Re^{ikL}} e^{ikL/3}$$



$$\left| \frac{A_t}{A_i} \right|^2 = \frac{(1-R)^2}{1+R^2+2R\cos(kL)} = \frac{(1-R)^2}{(1-R)^2 + 4R\cos^2(kL/2)}$$

A circular resonator

- The resonance frequencies depend on the cosine of the phase, not on the sine as in FP

$$I_t = I_i \frac{\left(1-R\right)^2}{\left(1-R\right)^2 + 4R\cos^2\left(kL/2\right)}$$

A circular resonator

- Thus the resonance frequencies are shifted, but the FSR is not changed:

$$\cos^2\left(kL/2\right)=0$$

$$\Rightarrow \frac{k_m L}{2}=\left(2m+1\right)\frac{\pi}{2}$$

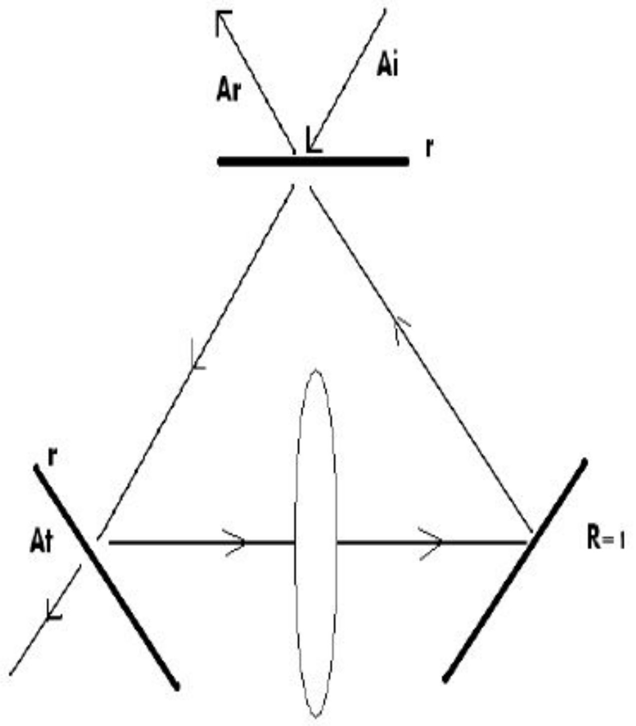
$$\Rightarrow \frac{2\pi nL}{2c}\nu_m=\left(2m+1\right)\frac{\pi}{2}$$

$$\Rightarrow \nu_m=\frac{c}{2nL}\left(2m+1\right)$$

$$\Rightarrow \Delta\nu=\frac{c}{nL}$$

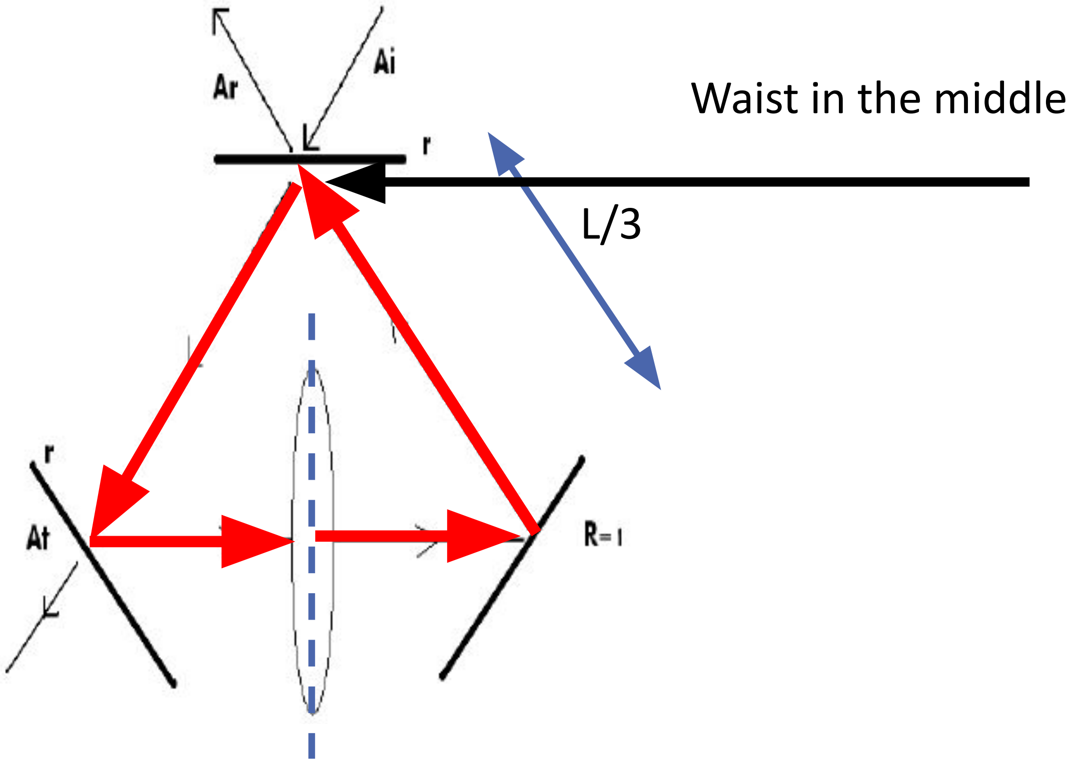
A circular resonator

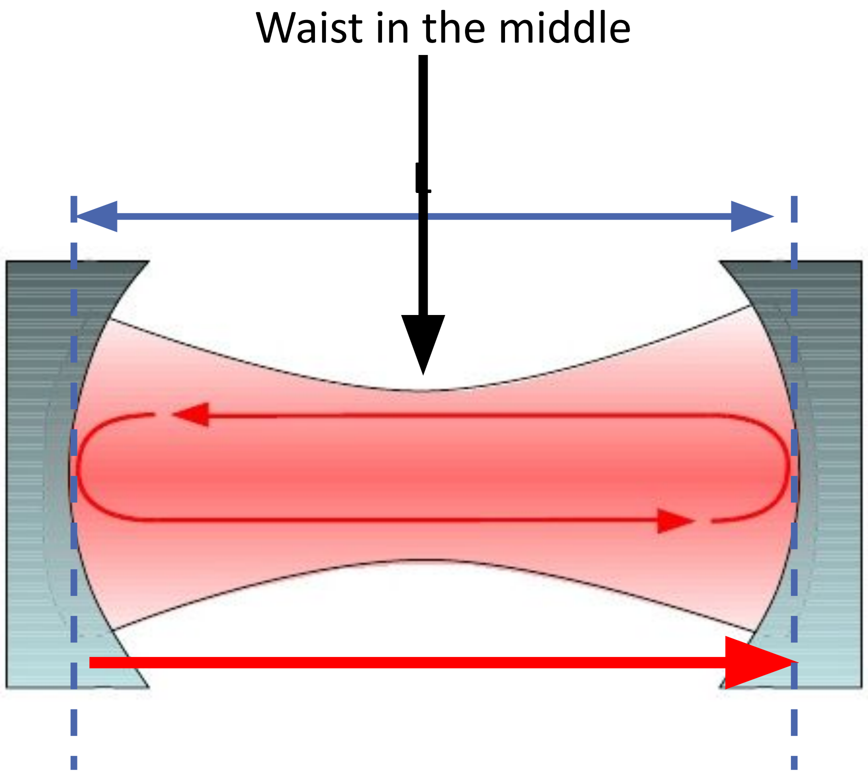
- **We add a mirror between the lower mirrors. Find the waist of the beam in the resonator**



A circular resonator

- We use the analogy to curved mirrors resonators:





A circular resonator

- We can calculate the size as in the curved mirrors resonator with *R=2f*:

$$z_2 = L/2 \quad R = 2f$$

$$\Rightarrow 2f = z_2 + \frac{z_0^{\,2}}{z_2} = \frac{L}{2} + 2\frac{z_0^{\,2}}{L}$$

$$\Rightarrow z_0^{\,2} = \frac{L}{2}\left[2f - \frac{L}{2}\right]$$

$$\omega_0^{\,2} = \frac{\lambda z_0 c}{\pi n} \qquad \Rightarrow \omega_0^{\,2} = \frac{\lambda}{\pi n}\sqrt{\frac{L}{2}\left[2f - \frac{L}{2}\right]}$$

A circular resonator

- **Find $\boldsymbol{v_{qlm}}$ for the first 6 modes for *f=L***

- We begin with finding the nonlinear phase from the relation of *L* and z_0

$$f = L \quad \Rightarrow z_0 = \frac{L}{2}\sqrt{3}$$

$$\Delta\eta = 2\tan^{-1}\left(\frac{L}{2z_0}\right) = 2\tan^{-1}\left(\frac{L}{2\frac{L}{2}\sqrt{3}}\right) \qquad \Rightarrow \Delta\eta = \frac{\pi}{3}$$

A circular resonator

- We notice that the output should gain a phase of some multiple of 2π

  over a distance L (not $2L$!)

- We should also add a π phase on each round due to reflection

A circular resonator

$$\frac{2\pi nL}{c}\nu_{qlm} - (l+m+1)\Delta\eta - \pi = 2q\pi$$

$$\Rightarrow \frac{2\pi nL}{c}\nu_{qlm} - (l+m+1)\frac{\pi}{3} = (2q+1)\pi$$

$$\Rightarrow \nu_{qlm} = \frac{c}{2nL}\left[(2q+1) + \frac{l+m+1}{3}\right]$$

A circular resonator

- The first 6 modes:

$$\nu_{q00} = \frac{c}{2nL}\left[1+\frac{1}{3}\right]+q\cdot FSR$$

$$\nu_{q01} = \nu_{q10} = \frac{c}{2nL}\left[1+\frac{2}{3}\right]+q\cdot FSR$$

$$\nu_{q11} = \nu_{q02} = \nu_{q20} = \frac{c}{2nL}\left[1+\frac{3}{3}\right]+q\cdot FSR$$

A circular resonator

- We see that the new resonances are shifted by c/2nL from FP (because of the π reflection phase)

$$\nu_{q00} = \frac{c}{2nL}\left[1+\frac{1}{3}\right]+q\cdot FSR$$

$$\nu_{q01} = \nu_{q10} = \frac{c}{2nL}\left[1+\frac{2}{3}\right]+q\cdot FSR$$

$$\nu_{q11} = \nu_{q02} = \nu_{q20} = \frac{c}{2nL}\left[1+\frac{3}{3}\right]+q\cdot FSR$$

- There are 5 new frequencies between each two: $\frac{l+m}{3}=2$