General problem of mathematical programming

$$f(y) \to \min, \; y \in Q \subseteq R^N \quad (1)$$

$$Q = \{y \in D : g_j(y) \le 0, 1 \le j \le m\} \quad (2)$$

$$D = \{y \in R^N : y_i \in [a_i, b_i], 1 \le i \le N\} \quad (3)$$

$$h(y) = 0 \sim |h(y)| \le \delta, \quad \delta > 0$$

$$h(y) = 0 \Leftrightarrow \begin{cases} h(y) \le 0, \\ h(\mathrm{y}) \ge 0, \end{cases}$$

$$\chi(Q) = \begin{cases} 0, \; y \in Q, \\ 1, \; y \notin Q, \end{cases}$$ - характеристическая функция множества Q

$$\chi(Q) \le 0$$

Auxiliary functions and sets

Feasibility function $G(y), y \in D:$

$$G(y) \le 0, y \in Q,$$
$$G(y) > 0, y \notin Q. \qquad (4)$$

$$G(y) = \max\{g_1(y),\ldots, g_m(y)\} \qquad (5)$$

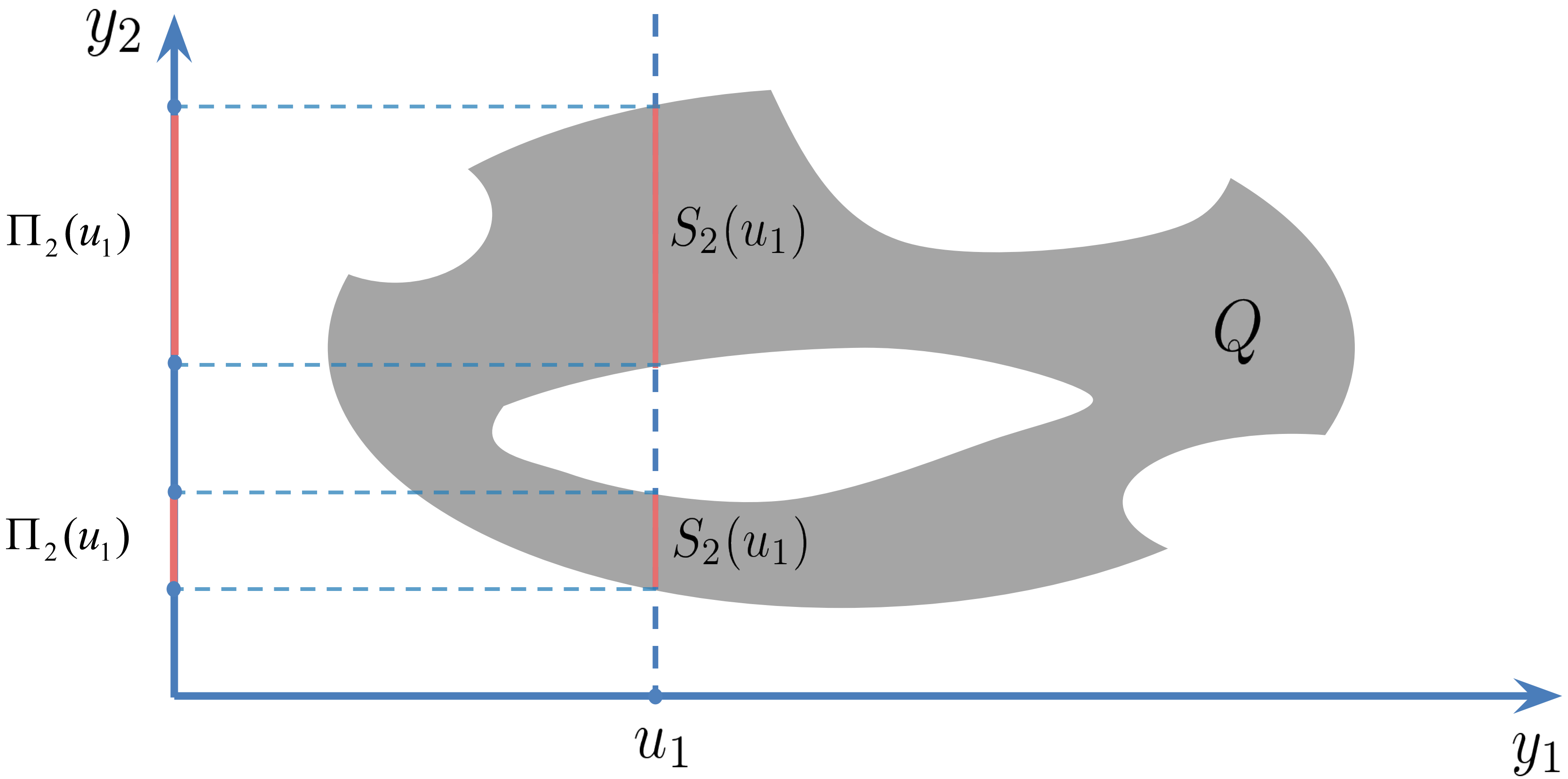
$$G(y) = \max\{0; g_1(y),\ldots, g_m(y)\} \qquad (6)$$

$$u_i = (y_1, \ldots, y_i), \quad v_i = (y_{i+1}, \ldots, y_N), 1 \le i \le N-1, \qquad (7)$$
$$u_N = y, i = N,$$
$$v_0 = y, i = 0.$$

$$S_1 = Q, \quad S_{i+1}(u_i) = \{v_i \in R^{N-i} : (u_i, v_i) \in Q\}, 1 \le i \le N-1 \qquad (8)$$

$$\Pi_{i+1}(u_i) = \{y_{i+1} \in R^1 : \exists (y_{i+1}, v_{i+1}) \in S_{i+1}(u_i)\} \qquad (9)$$

Sections and projections



Reducing the feasibility function

$$G^N(y) \equiv G(y)$$

(10)

$$D_i = \{u_i \in R^i : y_j \in [a_j, b_j], 1 \le j \le i\} \quad (11)$$

LEMMA 1

$$G^i(u_i) = \min\{G(u_i, v_i) : y_j \in [a_j, b_j], i+1 \le j \le N\} \quad (12)$$

Projecting Q onto coordinate axes $y_1, \ldots, y_i$ (13)

$$Q_i = \{u_i \in R^i : \exists (u_i, v_i) \in Q\}, 1 \le i \le N. \quad (14)$$

LEMMA 2

$$Q_i = \{u_i \in R^i : G^i(u_i) \le 0\} \quad (15)$$

LEMMA 3

$$\Pi_{i+1}(u_i) = \{y_{i+1} \in [a_{i+1}, b_{i+1}] : G^{i+1}(u_i, y_{i+1}) \le 0\} \quad (16)$$

Nested optimization scheme

Nested optimization scheme

$$f^N(y) \equiv f(y)$$

$$f^i(u_i) = \min\{f^{i+1}(u_i, y_{i+1}) : y_{i+1} \in \Pi_{i+1}(u_i)\}, u_i \in Q_i \tag{17}$$

Main relation

$$\min_{y \in Q} f(y) = \min_{y_1 \in \Pi_1} \min_{y_2 \in \Pi_2(u_1)} \dots \min_{y_N \in \Pi_N(u_{N-1})} f(y) \tag{18}$$

$$f^1(y_1) \to \min, \ y_1 \in \Pi_1 \subseteq R^1$$

$$\Pi_1 = \{y_1 \in [a_1, b_1] : G^1(y_1) \le 0\}$$

$$f^2(y_1, y_2) \to \min, \ y_2 \in \Pi_2(y_1) \subseteq R^1,$$

$$\Pi_2(y_1) = \{y_2 \in [a_2, b_2] : G^2(y_1, y_2) \le 0\}$$

$$f^3(u_2, y_3) \to \min$$

$$\cdot \quad \cdot \quad \cdot$$

$$f^N(u_{N-1}, y_N) = f(u_{N-1}, y_N) \to \min, \ y_N \in \Pi_N(u_{N-1}).$$

Nested optimization scheme

$$f^i(u_{i-1}, y_i) \to \min,\ y_i \in \Pi_i(u_{i-1}) \qquad (19)$$

$$u_{i-1} \in Q_{i-1} - \text{fixed}$$

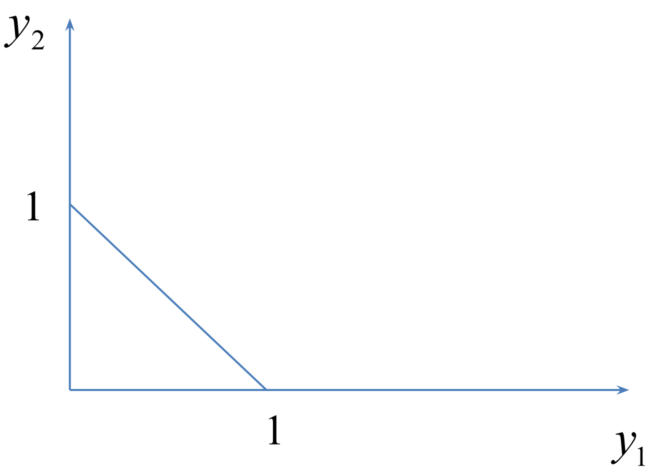
$$\underset{y_1 \in \Pi_1}{extr}\ \underset{y_2 \in \Pi_2(u_1)}{extr} \dots \underset{y_N \in \Pi_N(u_{N-1})}{extr} \varphi(y)$$

Example

$$f(y_1, y_2) = (y_1 - 1)^2 + (y_2 - 1)^2$$

$$g_1(y) = y_1 + y_2 - 1$$

$$0 \le y_1 \le 2, 0 \le y_2 \le 2$$



Nested optimization scheme

$$G^2(y) = y_1 + y_2 - 1$$

$$G^1(y_1) = \min\{y_1 + y_2 - 1, y_2 \in [0, 2]\} = y_1 - 1$$

$$\Pi_2(y_1) = \{y_2 \in [0, 2] : y_1 + y_2 - 1 \le 0\} = [0, 1 - y_1]$$

$$\Pi_1 = \{y_1 \in [0,2] : y_1 - 1 \le 0\} = [0, 1]$$

$$f^2(y_1, y_2) = (y_1 - 1)^2 + (y_2 - 1)^2$$

$$f^1(y_1) = \min\{(y_1 - 1)^2 + (y_2 - 1)^2 : y_2 \in \Pi_2(y_1) = [0, 1 - y_1]\}$$

Функция $(y_1 - 1)^2 + (y_2 - 1)^2$ достигает минимума по y_2 в единице, а слева от 1 убывает, поэтому на отрезке $[0, 1 - y_1]$ ее минимум достигается в точке $y_2 = 1 - y_1$, а значение равно $2y_1^2 - 2y_1 + 1$, т.е.

$$f^1(y_1) = 2y_1^2 - 2y_1 + 1$$

Ее минимум на отрезке [0,1] достигается в точке $y_1^* = 1/2$

$f^2(0.5, y_2) = 1/4 + (y_2 - 1)^2$ Т.к. $\Pi_2(1/2) = [0, 1 - 1/2] = [0, 1/2]$, где f^2 убывает, то $y_2^* = 1/2$