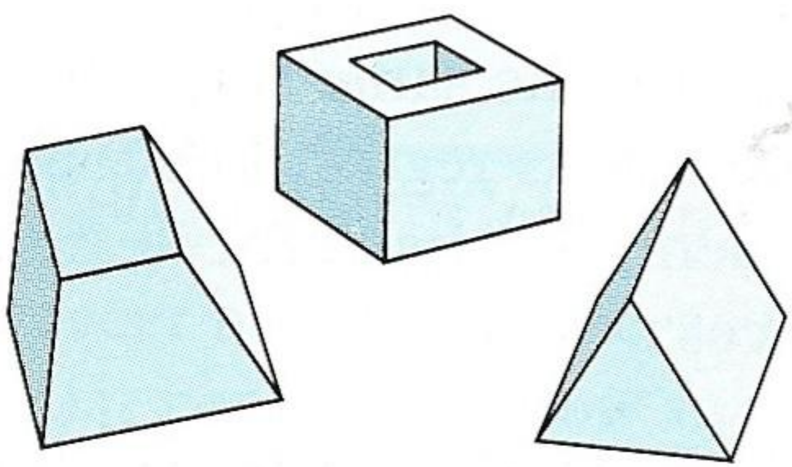
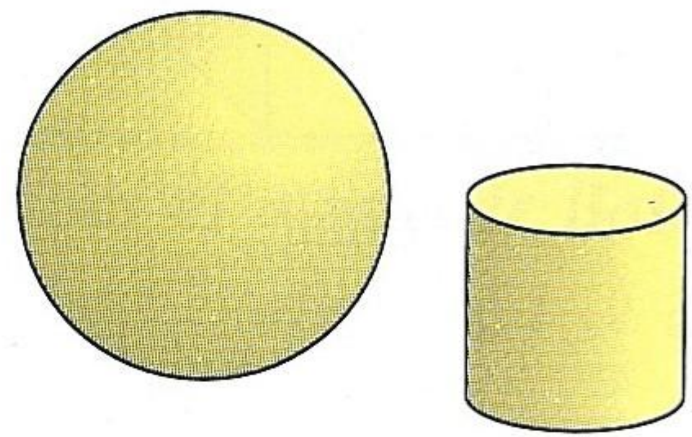
# volumes

# Polyhedrons

## What is a polyhedron?



These are polyhedrons.



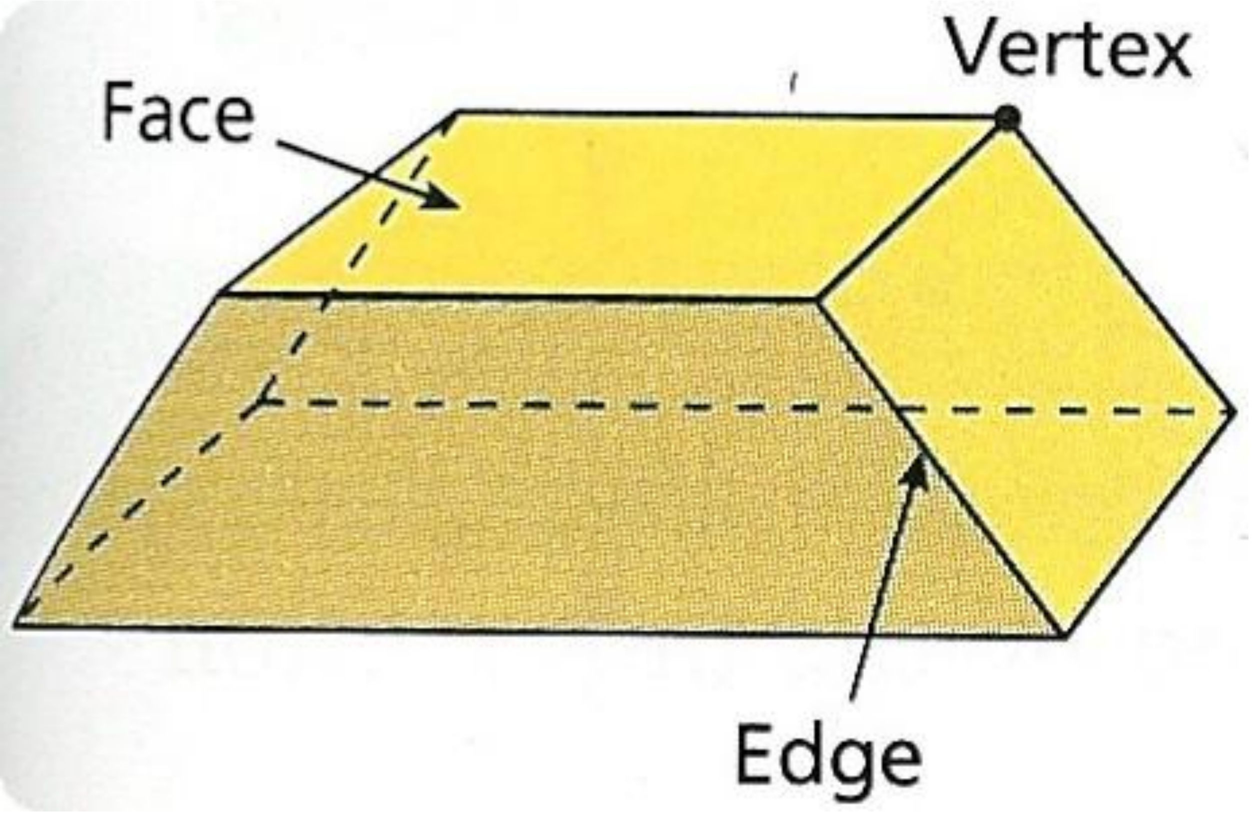
These are not polyhedrons.

Circles are not polygons

# Identifying Polyhedrons

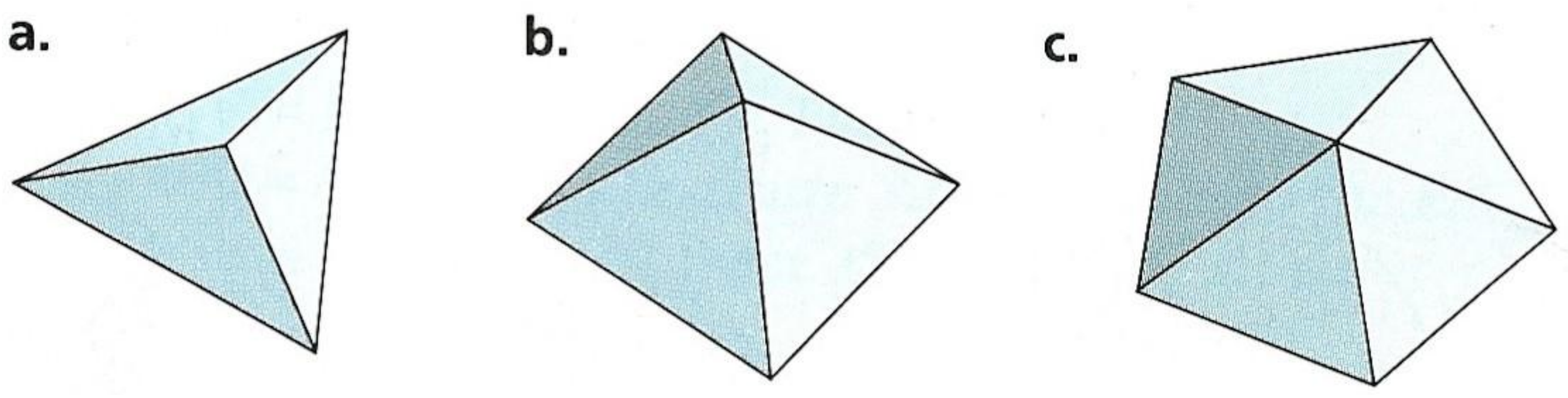
- A **polyhedron** is a solid that is bounded by polygons, called ***faces***, that enclose a single region of space.
- An ***edge*** of polyhedron is a line segment formed by the intersection of two faces
- A **vertex** of a polyhedron is a point where three or more edges meet

# Parts of a Polyhedron



# Example 1
## Counting Faces, Vertices, and Edges

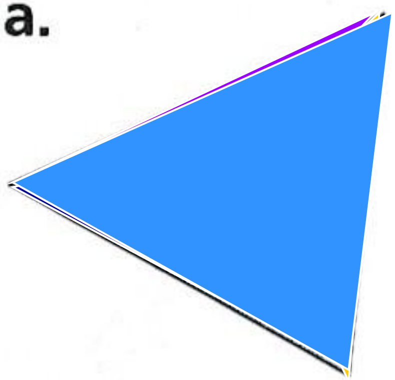
- Count the faces, vertices, and edges of each polyhedron



# Example 1A
# Counting Faces

- Count the faces, vertices, and edges of each polyhedron

a.

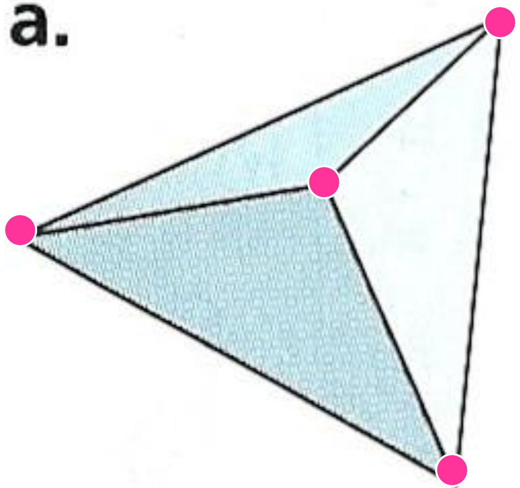


4 faces

# Example 1a
# Counting Vertices

- Count the faces, vertices, and edges of each polyhedron

a.

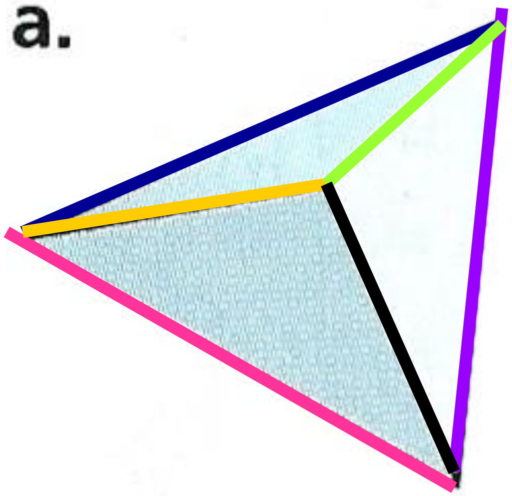


4 vertices

# Example 1a
# Counting Edges

- Count the faces, vertices, and edges of each polyhedron

a.

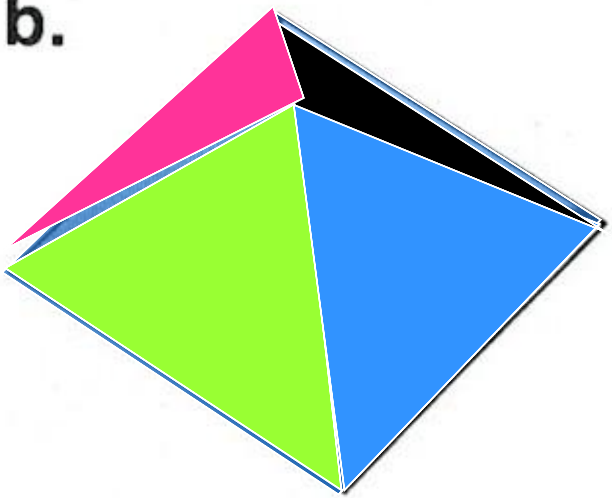


6 edges

# Example 1b
# Counting Faces

- Count the faces, vertices, and edges of each polyhedron

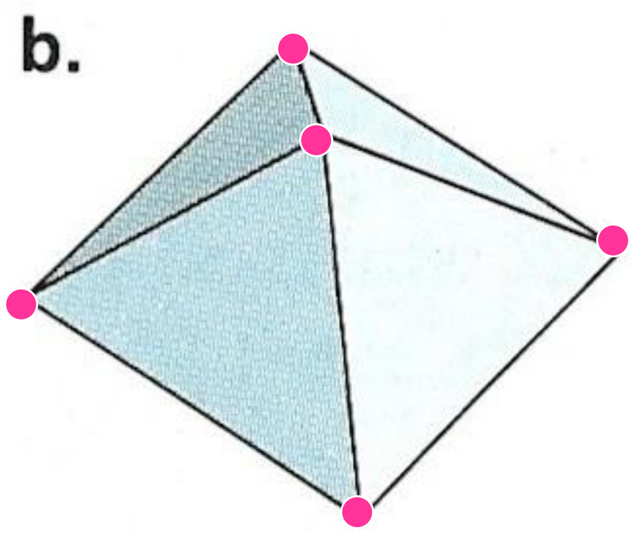
b.



5 faces

# Example 1b
# Counting Vertices

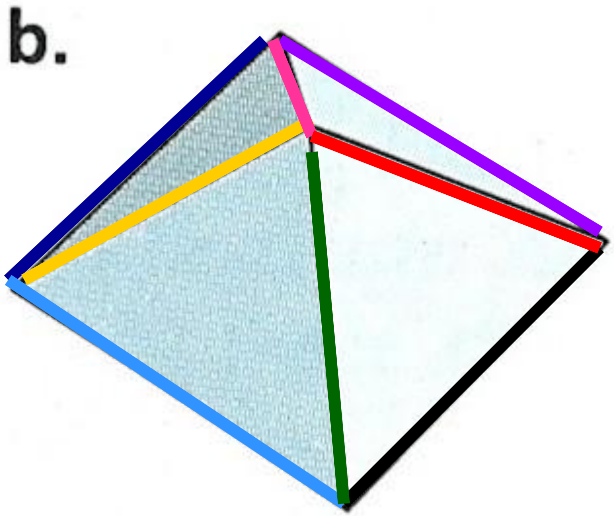
- Count the faces, vertices, and edges of each polyhedron




5 vertices

# Example 1b
## Counting Vertices

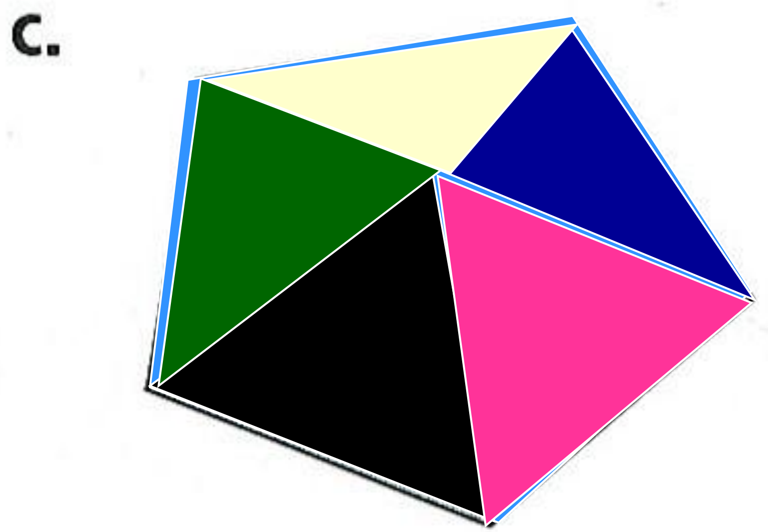
- Count the faces, vertices, and edges of each polyhedron




8 edges

# Example 1c
# Counting Faces

- Count the faces, vertices, and edges of each polyhedron

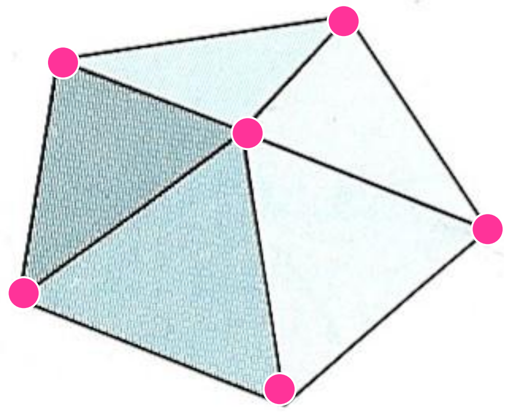



6 faces

# Example 1c
# Counting Vertices

- Count the faces, vertices, and edges of each polyhedron

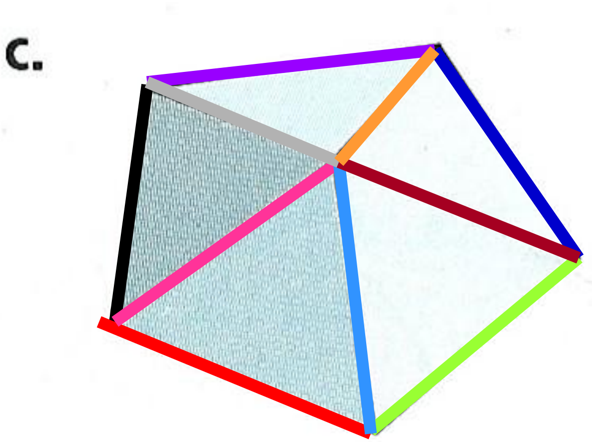
c.



6 vertices

# Example 1c
# Counting Edges

- Count the faces, vertices, and edges of each polyhedron




10 edges

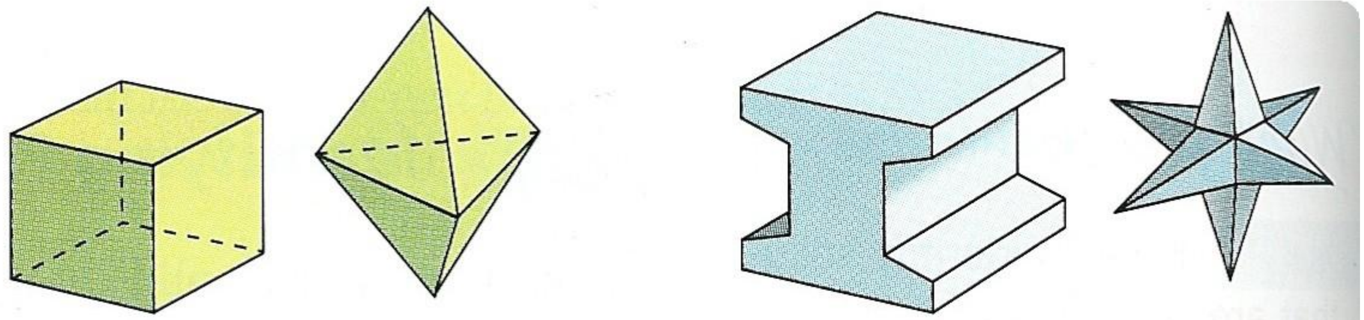
# Notice a Pattern?

| Faces | Vertices | Edges |
|---|---|---|
| 4 | 4 | 6 |
| 5 | 5 | 8 |
| 6 | 6 | 10 |

# Theorem 12.1
# Euler's Theorem

- The number of faces (F), vertices (V), and edges (E) of a polyhedron is related by $F + V = E + 2$

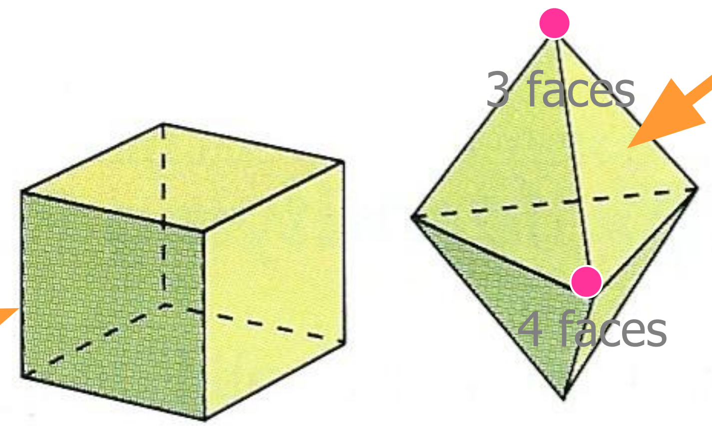
- More about Polyhedrons
  - The **surface** of a polyhedron consists of all points on its faces
  - A polyhedron is **convex** if any two points on its surface can be connected by a line segment that lies entirely inside or on the polyhedron



*These polyhedrons are convex.* *These polyhedrons are not convex.*

# Regular Polyhedrons

- A polyhedron is **regular** if all its faces are congruent regular polygons.



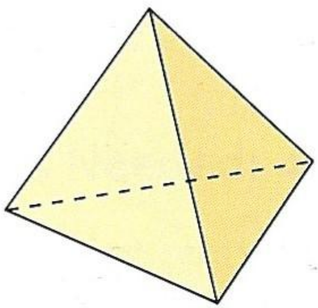

*These polyhedrons are convex.*

regular

Not regular

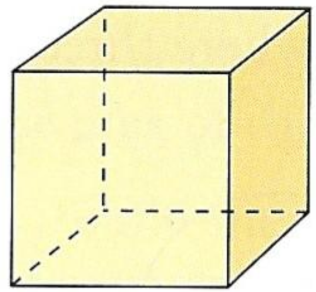
Vertices are not formed by the same number of faces

# 5 kinds of Regular Polyhedrons



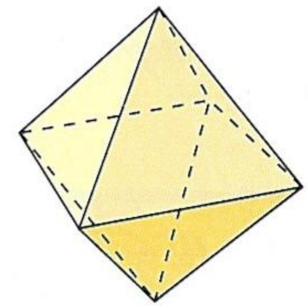
*Regular tetrahedron*

4 faces



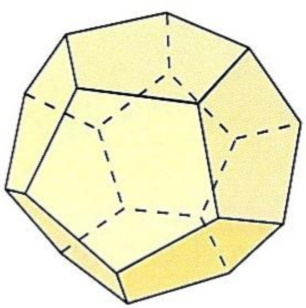
*Cube*

6 faces



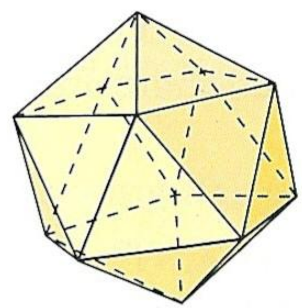
*Regular octahedron*

8 faces



*Regular dodecahedron*

12 faces

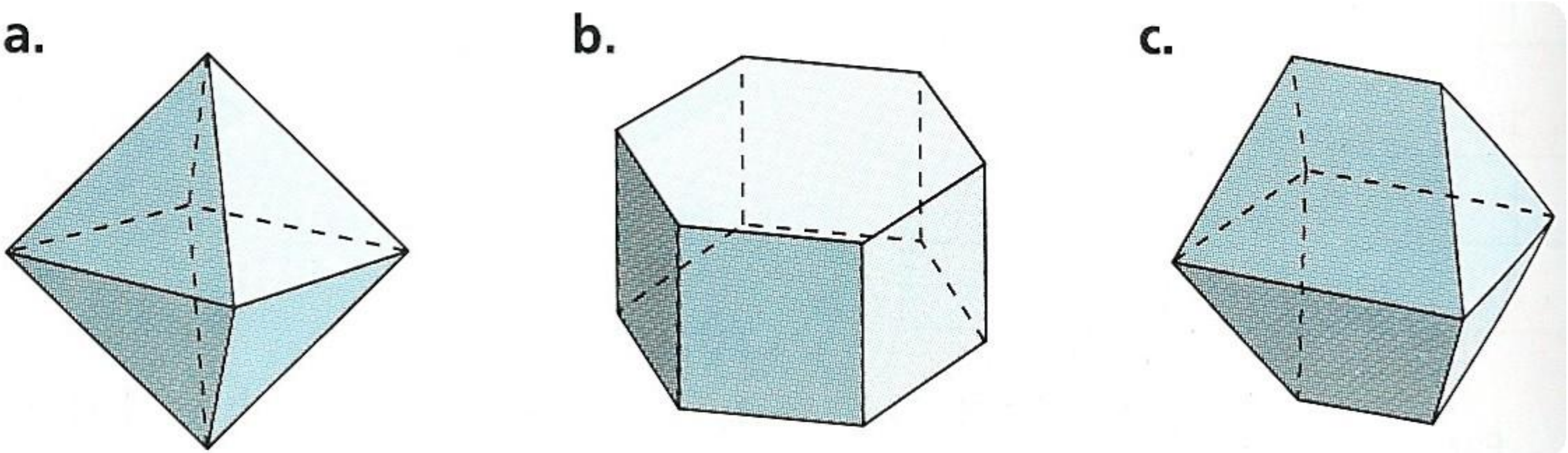


*Regular icosahedron*

20 faces

# Example 2
# Classifying Polyhedrons

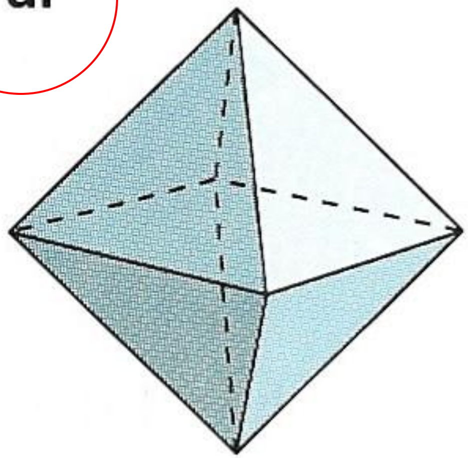
- One of the octahedrons is regular. Which is it?




A polyhedron is **regular** if all its faces are congruent regular polygons.

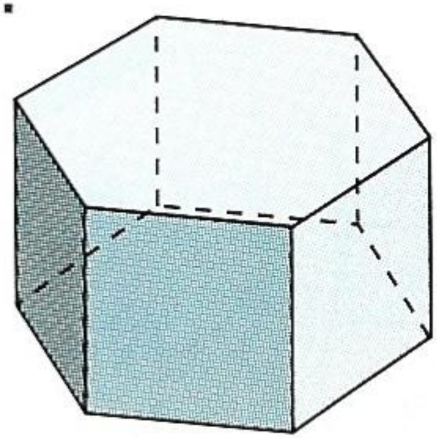
# Example 2
# Classifying Polyhedrons

a.



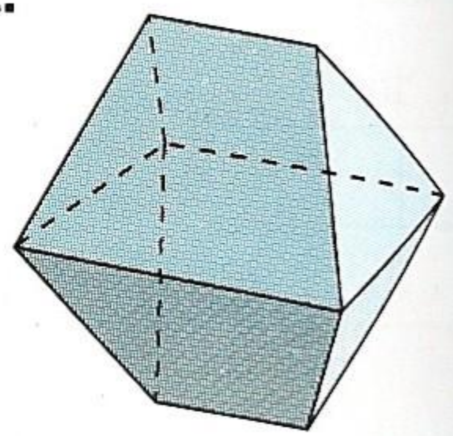
All its faces are congruent equilateral triangles, and each vertex is formed by the intersection of 4 faces

b.



Faces are not all congruent (regular hexagons and squares)

c.



Faces are not all regular polygons or congruent (trapezoids and triangles)

# Example 3
## Counting the Vertices of a Soccer Ball

- A soccer ball has 32 faces: 20 are regular hexagons and 12 are regular pentagons. How many vertices does it have?
-

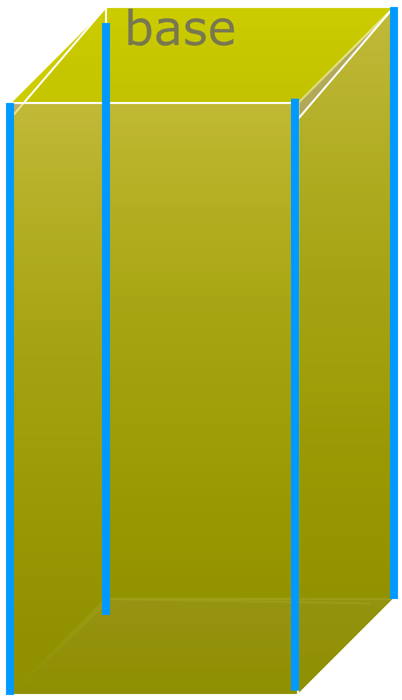
  - A soccer ball is an example of a semiregular polyhedron - one whose faces are more than one type of regular polygon and whose vertices are all exactly the same

# Example 3
# Counting the Vertices of a Soccer Ball

- A soccer ball has 32 faces: 20 are regular hexagons and 12 are regular pentagons. How many vertices does it have?
- Hexagon = 6 sides, Pentagon = 5 sides
- Each edge of the soccer ball is shared by two sides
- Total number of edges = $\frac{1}{2}(6 \bullet 20 + 5 \bullet 12) = \frac{1}{2}(180) = 90$
- Now use Euler's Theorem
- $F + V = E + 2$
- $32 + V = 90 + 2$
- $V = 60$

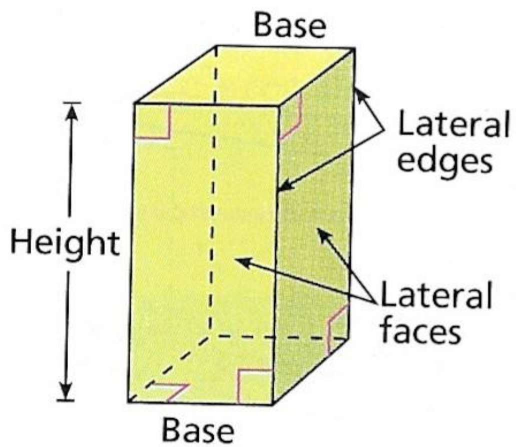
# Prisms

- A **prism** is a polyhedron that has two parallel, congruent faces called *bases*.
- The other faces, called *lateral faces*, are parallelograms and are formed by connecting corresponding vertices of the bases
- The segment connecting these corresponding vertices are *lateral edges*
- Prisms are classified by their bases

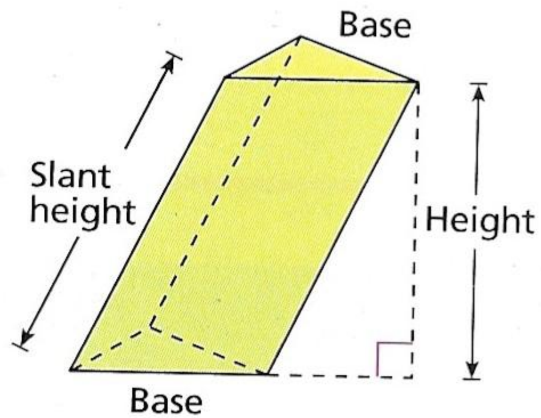


# Prisms

- The *altitude* or *height*, of a prism is the perpendicular distance between its bases
- In a **right prism**, each lateral edge is perpendicular to both bases
- Prisms that have lateral edges that are oblique (**≠90°)** to the bases are **oblique prisms**
- The length of the oblique lateral edges is the *slant height* of the prism




***Right rectangular prism***




***Oblique triangular prism***

# Surface Area of a Prism

- The surface area of a polyhedron is the sum of the areas of its faces

# Example 1
# Find the Surface Area of a Prism

- The Skyscraper is 414 meters high. The base is a square with sides that are 64 meters. What is the surface area of the skyscraper?

# Example 1
# Find the Surface Area of a Prism

- The Skyscraper is 414 meters high. The base is a square with sides that are 64 meters. What is the surface area of the skyscraper?

$64(64)=4096$

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$64(414)=26496$

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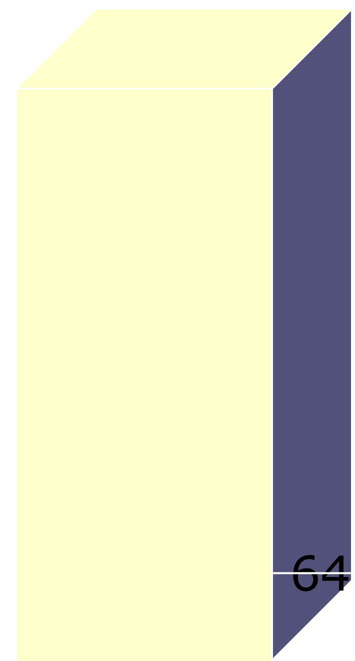



Surface Area = $4(64 \bullet 414)+2(64 \bullet 64)=114{,}176\ m^2$

# Example 1
# Find the Surface Area of a Prism

- The Skyscraper is 414 meters high. The base is a square with sides that are 64 meters. What is the surface area of the skyscraper?

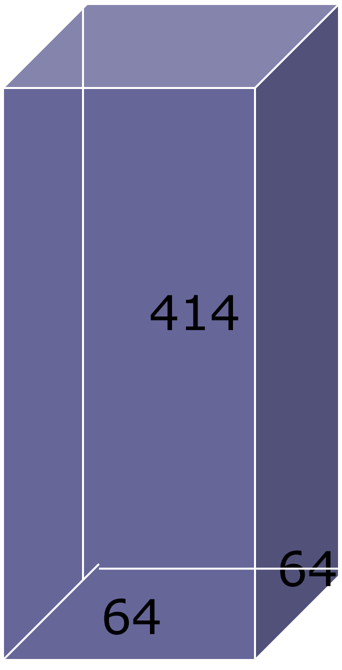
Surface Area = $4(64 \bullet 414)+2(64 \bullet 64)=114{,}176\ m^2$

Surface Area = $(4 \bullet 64) \bullet 414+2(64 \bullet 64)=114{,}176\ m^2$

height

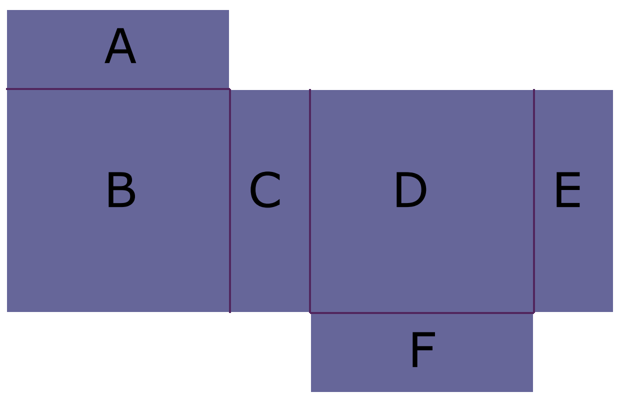
Perimeter of the base

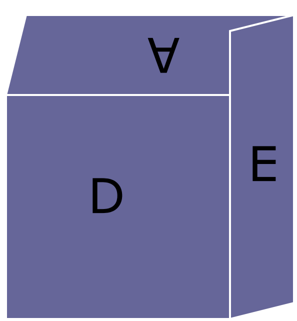
Area of the base



# Nets

- A net is a pattern that can be cut and folded to form a polyhedron.



# Surface Area of a Right Prism

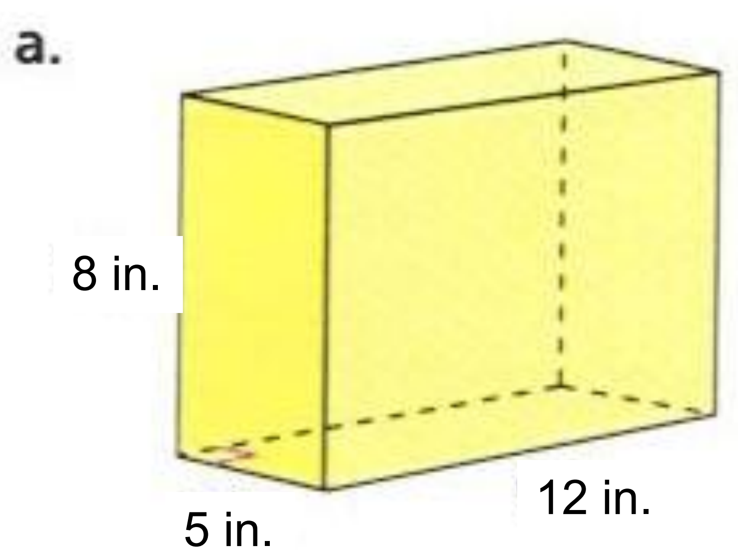
- The surface area, S, of a right prism is

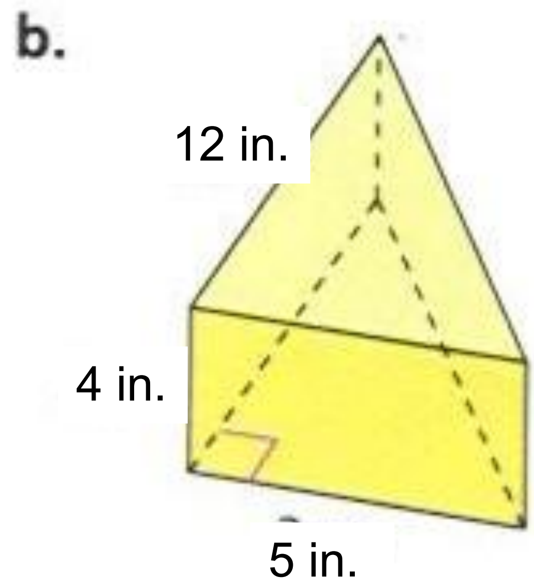
  $S = 2B + Ph$

  where B is the area of a base, P is the perimeter of a base, and h is the height

# Example 2
# Finding the Surface Area of a Prism

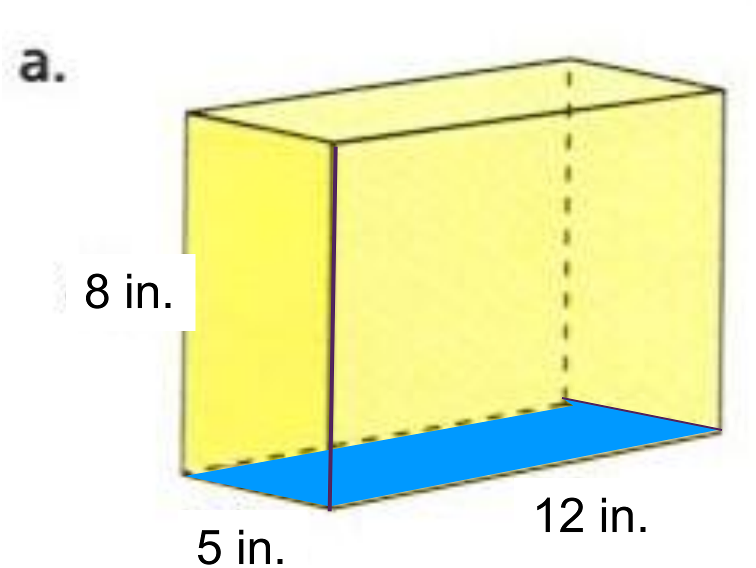
- Find the surface area of each right prism



# Example 2
# Finding the Surface Area of a Prism

- Find the surface area of each right prism




$S = 2B + Ph$

Area of the Base = 5x12=60

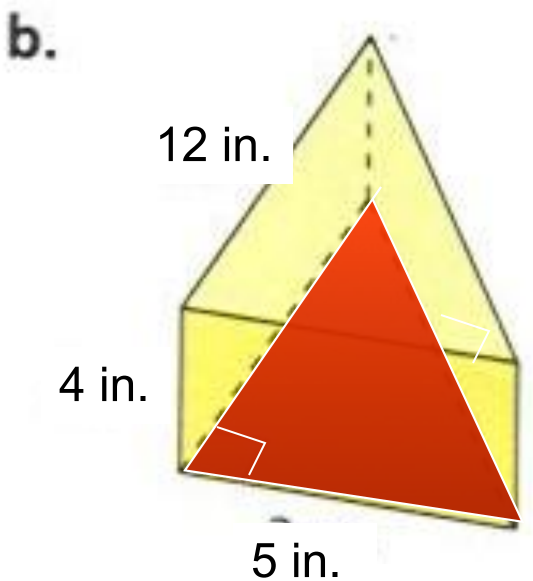
Perimeter of Base = 5+12+5+12 = 34

Height of Prism = 8

- $S = 2B + Ph$
- $S = 2(60) + (34) \bullet 8$
- $S = 120 + 272 = 392\ in^2$

# Example 2
# Finding the Surface Area of a Prism

- Find the surface area of each right prism




$S = 2B + Ph$

Area of the Base = ½(5)(12)=30

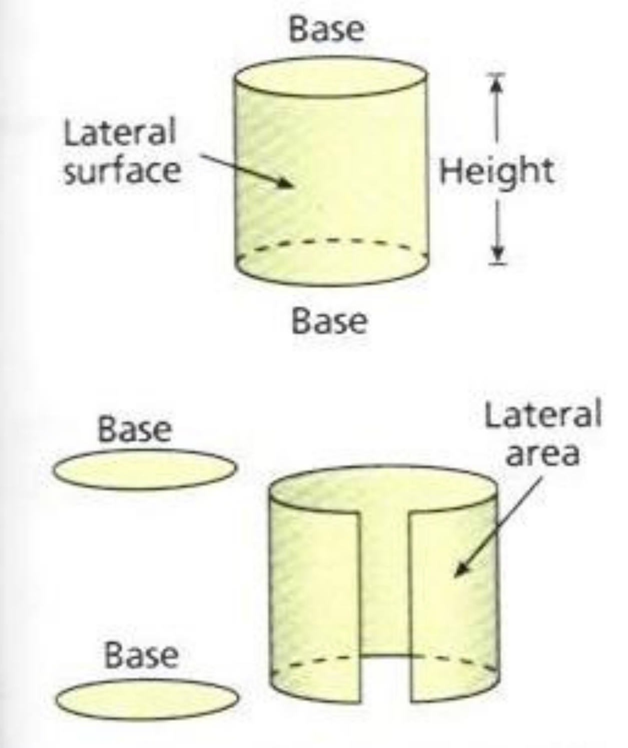
Perimeter of Base = 5+12+13=30

Height of Prism = 4
(distance between triangles)

- $S = 2B + Ph$
- $S = 2(30) + (30) \bullet 4$
- $S = 60 + 120 = 180 \text{ in}^2$

# Cylinders

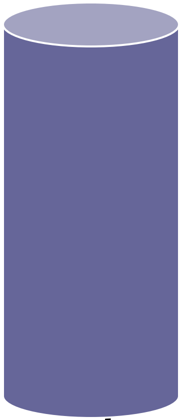
- A **cylinder** is a solid with congruent circular bases that lie in parallel planes
- The altitude, or height, of a cylinder is the perpendicular distance between its bases
- The **lateral area** of a cylinder is the area of its curved lateral surface.
- A cylinder is **right** if the segment joining the centers of its bases is perpendicular to its bases




*The surface area of a right cylinder is the sum of the areas of its two bases and its lateral area.*

# Surface Area of a Right Cylinder

The surface area, S, of a right circular cylinder is



$S = 2B + Ch$

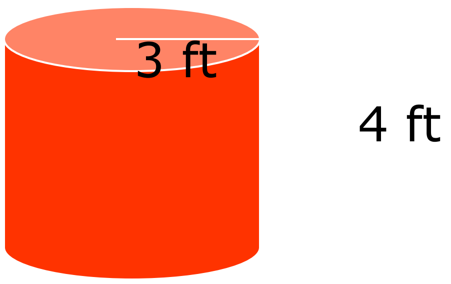
or

$2\pi r^2 + 2\pi rh$

where B is the area of a base, C is the circumference of a base, r is the radius of a base, and h is the height

# Example 3
# Finding the Surface Area of a Cylinder

- Find the surface area of the cylinder

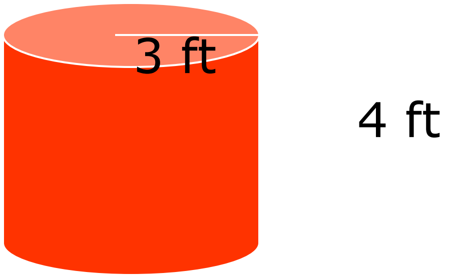


# Example 3
# Finding the Surface Area of a Cylinder

- Find the surface area of the cylinder
- $2\pi r^2 + 2\pi rh$
- $2\pi(3)^2 + 2\pi(3)(4)$
- $18\pi + 24\pi$
- $42\pi \approx 131.9 \text{ ft}^2$

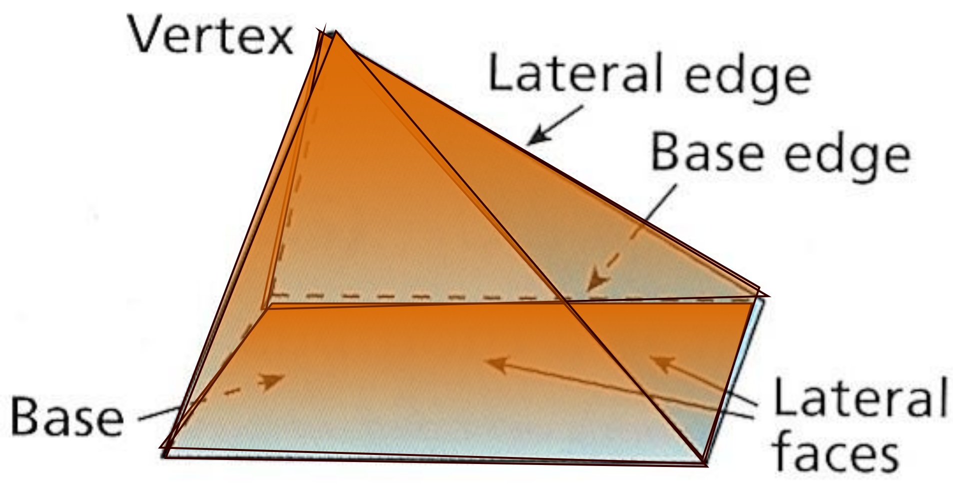
Radius = 3

Height = 4



# Pyramids

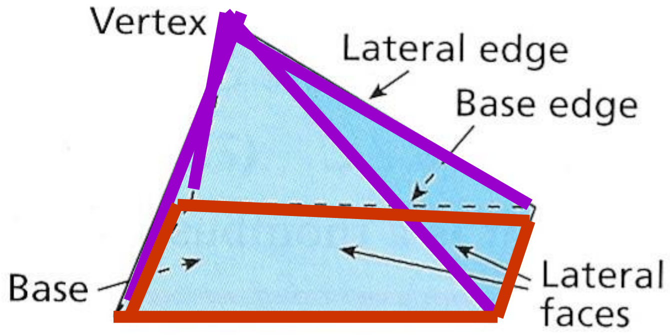
- A pyramid is a polyhedron in which the *base* is a polygon and the *lateral faces* are triangles that have a common vertex



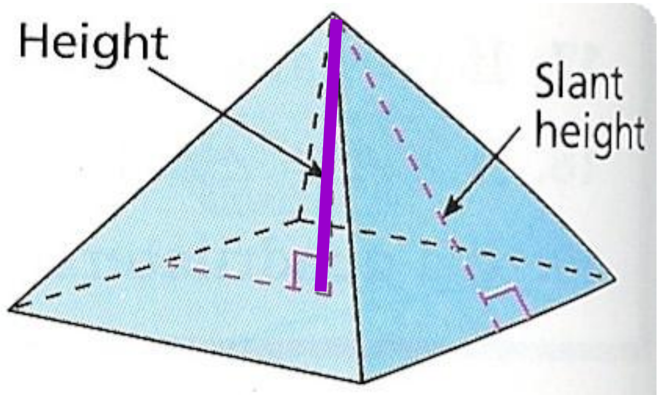

***Pyramid***

# Pyramids

- The intersection of two lateral faces is a *lateral edge*
- The intersection of the base and a lateral face is a *base edge*
- The *altitude* or *height* of the pyramid is the perpendicular distance between the base and the vertex



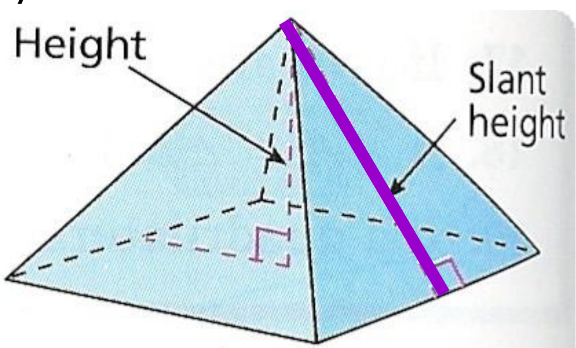

*Pyramid*



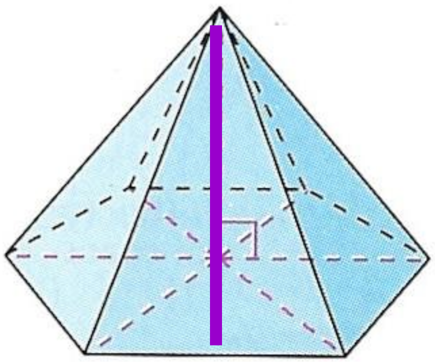

*Regular pyramid*

# Regular Pyramid

- A pyramid is regular if its base is a regular polygon and if the segment from the vertex to the center of the base is perpendicular to the base
- The slant height of a regular pyramid is the altitude of any lateral face (a nonregular pyramid has no slant height)



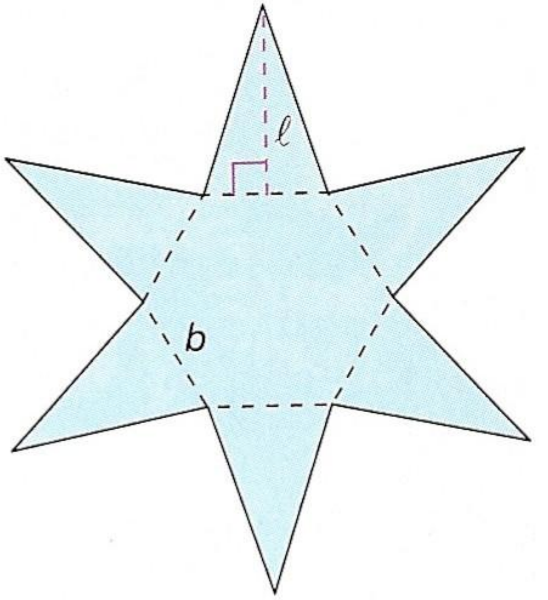

Regular pyramid



Regular pyramid

# Developing the formula for surface area of a regular pyramid

- Area of each triangle is ½bL
- Perimeter of the base is 6b
- Surface Area =
  (Area of base) + 6(Area of lateral faces)
- $S = B + 6(\frac{1}{2}b\mathbf{l})$
- $S = B + \frac{1}{2}(6b)(\mathbf{l})$
- $S = B + \frac{1}{2}P\mathbf{l}$



# Surface Area of a Regular Pyramid

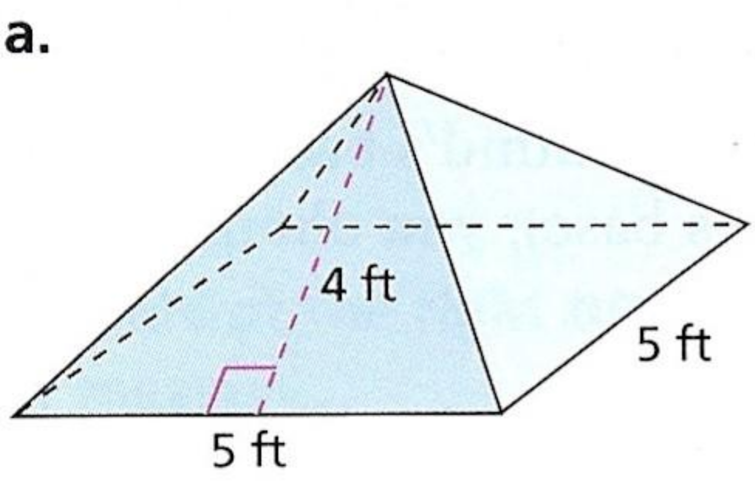
- The surface area, S, of a regular pyramid is

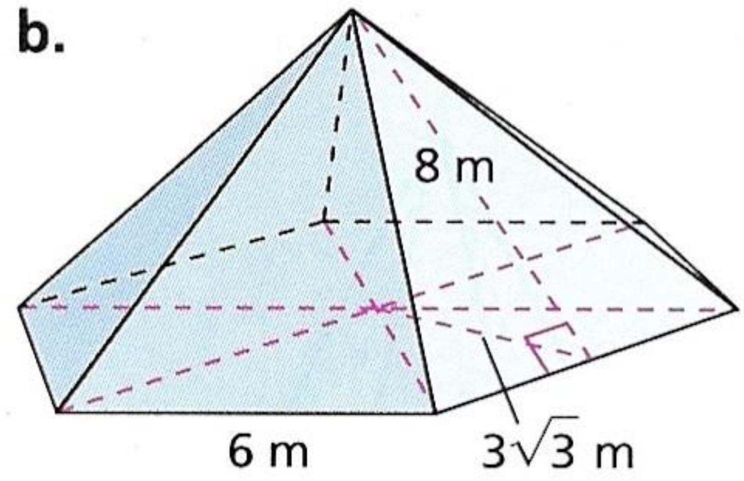
  $S = B + \frac{1}{2}P\ell$

  Where B is the area of the base, P is the perimeter of the base, and L is the slant height

# Example 1
# Finding the Surface Area of a Pyramid

- Find the surface area of each regular pyramid

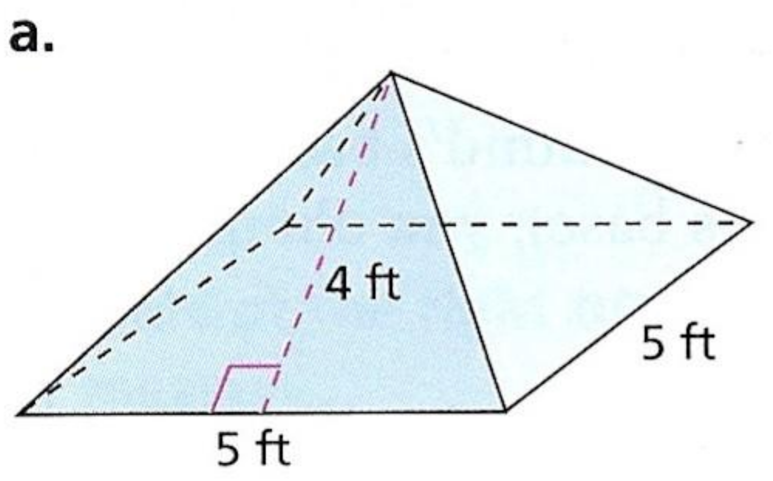


# Example 1
# Finding the Surface Area of a Pyramid

- Find the surface area of each regular pyramid.

$S = B + \frac{1}{2}PL$




Base is a Square

Area of Base = 5(5) = 25

Perimeter of Base

5+5+5+5 = 20

Slant Height = 4

$S = 25 + \frac{1}{2}(20)(4)$

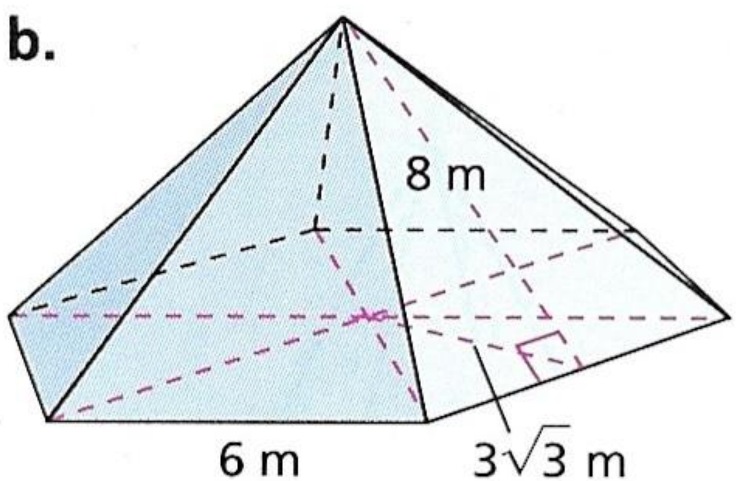
$= 25 + 40$

$= 65 \text{ ft}^2$

# Example 1
# Finding the Surface Area of a Pyramid

- Find the surface area of each regular pyramid

S = B + ½PL




Base is a Hexagon

A=½aP

$$A = \frac{1}{2}(3\sqrt{3})(36) = 54\sqrt{3}$$

Perimeter = 6(6)=36

Slant Height = 8

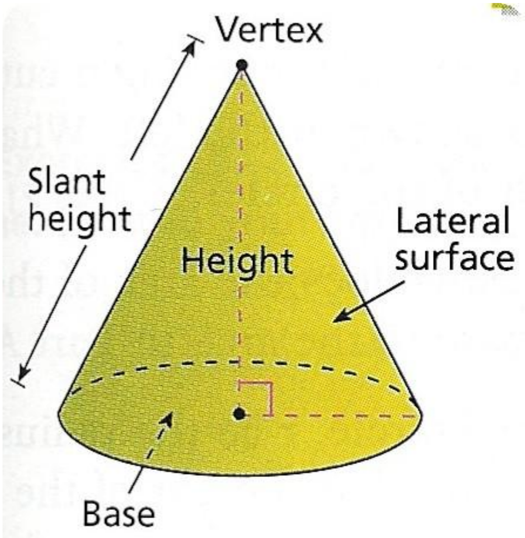
S = $54\sqrt{3}$ + ½(36)(8)

= $54\sqrt{3}$ + 144

= 237.5 $m^2$

# Cones

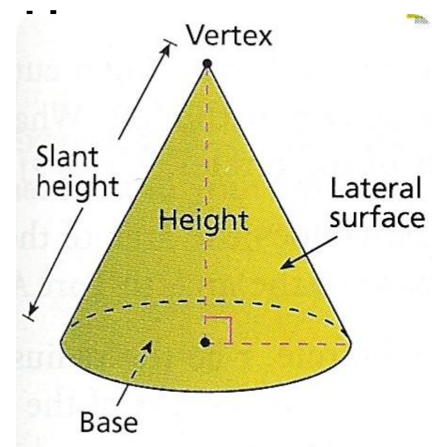
- A **cone** is a solid that has a circular base and a *vertex* that is not in the same plane as the base
- The *lateral surface* consists of all segments that connect the vertex with point on the edge of the base
- The *altitude*, or *height*, of a cone is the perpendicular distance between the vertex and the plane that contains the base



*Right cone*

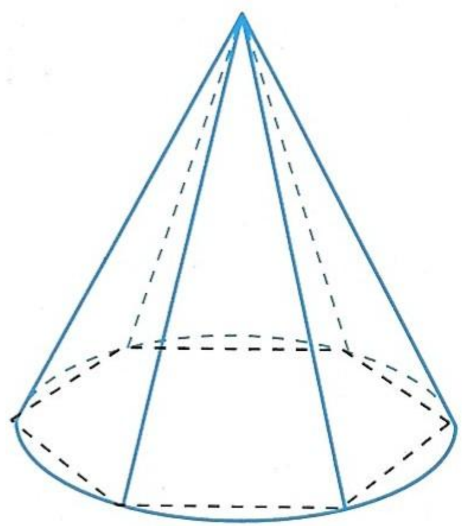
# Right Cone

- A right cone is one in which the vertex lies directly above the center of the base
- The slant height of a right cone is the distance between the vertex and a point on the edge of base




*Right cone*

# Developing the formula for the surface area of a right cone



- Use the formula for surface area of a pyramid $S = B + \frac{1}{2}P\ell$
- As the number of sides on the base increase it becomes nearly circular
- Replace ½P (half the perimeter of the pyramids base) with $\pi r$ (half the circumference of the cone's base)

# Surface Area of a Right Cone

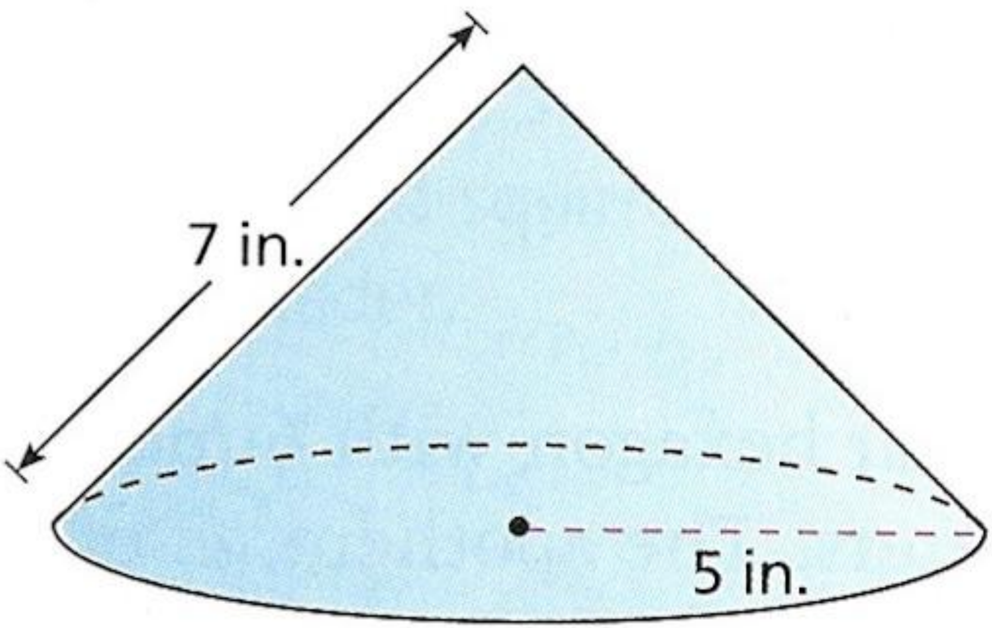
The surface area, S, of a right cone is

$$S = \pi r^2 + \pi r\ell$$

Where r is the radius of the base and L is the slant height of the cone

# Example 2
## Finding the Surface Area of a Right Cone

Find the surface area of the right cone



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## Finding the Surface Area of a Right Cone

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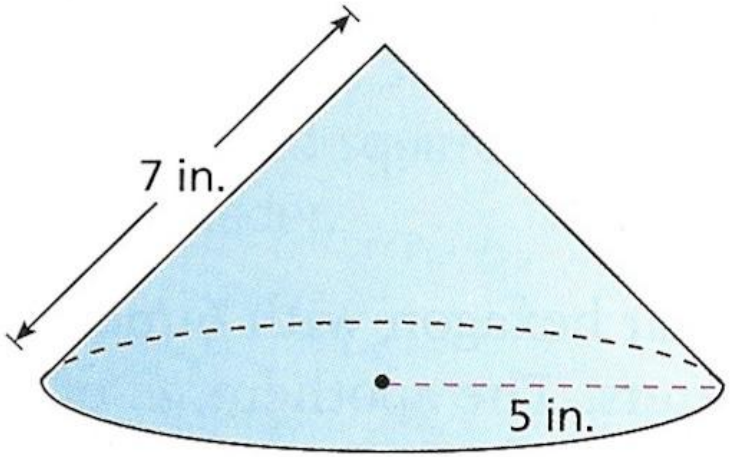



$$S = \pi r^2 + \pi r\ell$$
$$= \pi(5)^2 + \pi(5)(7)$$
$$= 25\pi + 35\pi$$
$$= 60\pi \text{ or } 188.5 \text{ in}^2$$

Radius = 5

Slant height = 7

# Volume formulas

The Volume, V, of a prism is $V = Bh$

The Volume, V, of a cylinder is $V = \pi r^2 h$

The Volume, V, of a pyramid is $V = {}^{1}/_{3}Bh$

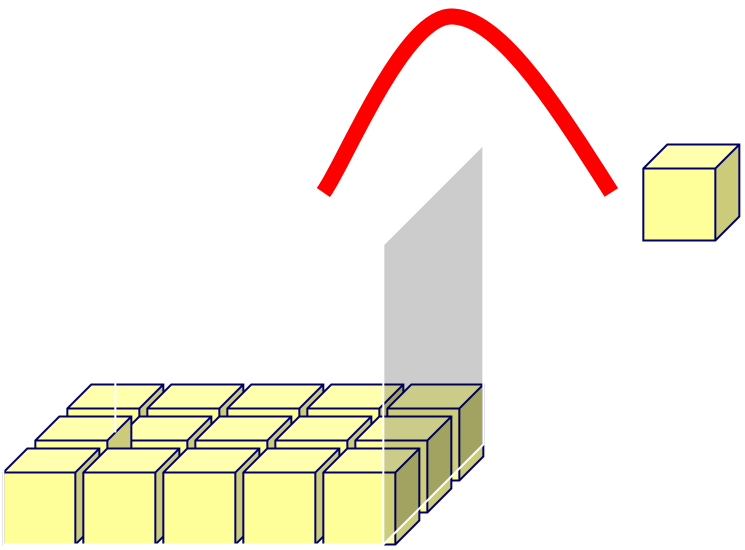
The Volume, V, of a cone is $V = {}^{1}/_{3}\pi r^2 h$

The Surface Area, S, of a sphere is $S = 4\pi r^2$

The Volume, V, of a sphere is $V = {}^{4}/_{3}\pi r^3$

# Volume

- The volume of a polyhedron is the number of cubic units contained in its interior
- Label volumes in cubic units like $cm^3$, $in^3$, $ft^3$, etc

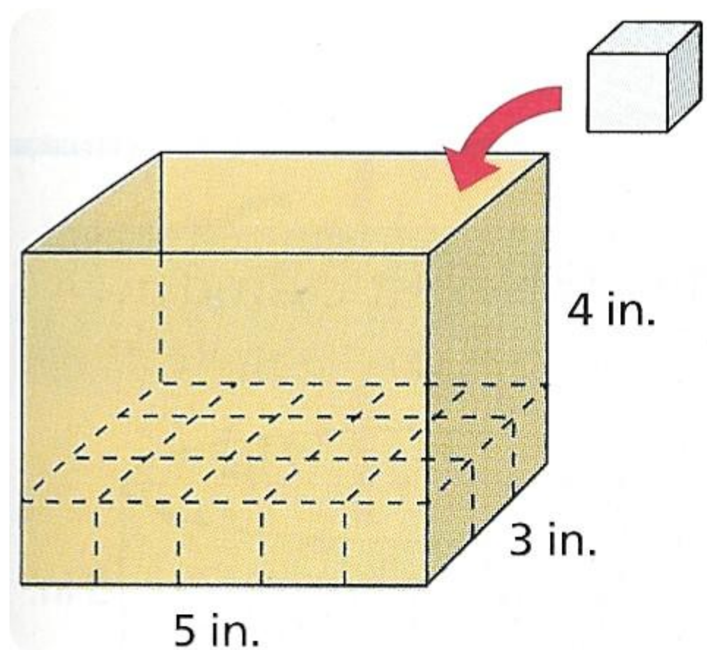


# Postulates

- All the formulas for the volumes of polyhedrons are based on the following three postulates
  - Volume of Cube Postulate: The volume of a cube is the cube of the length of its side, or $V = s^3$
  - Volume Congruence Postulate: If two polyhedrons are congruent, then they have the same volume
  - Volume Addition Postulate: The volume of a solid is the sum of the volumes of all its nonoverlapping parts

# Example 1: Finding the Volume of a Rectangular Prism

- The cardboard box is 5" x 3" x 4" How many unit cubes can be packed into the box? What is the volume of the box?



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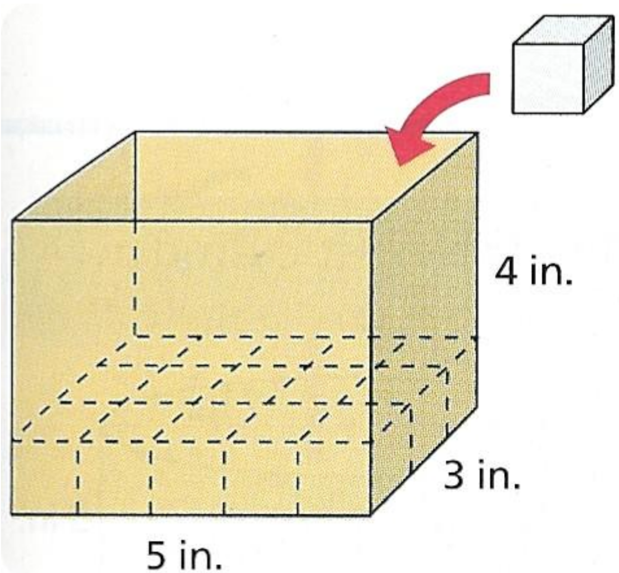


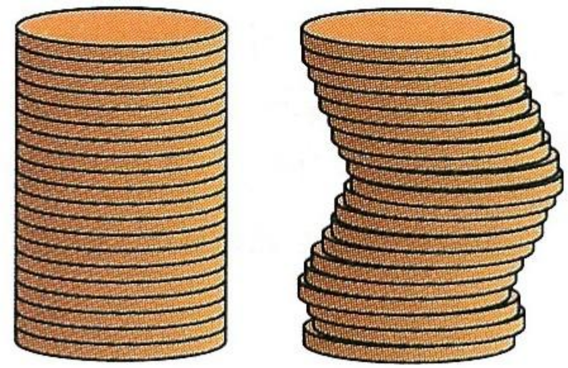

- How many cubes in bottom layer?
- 5(3) = 15
- How many layers?
- 4
- V=5(3)(4) = 60 in$^3$

V = L x W x H for a rectangular prism

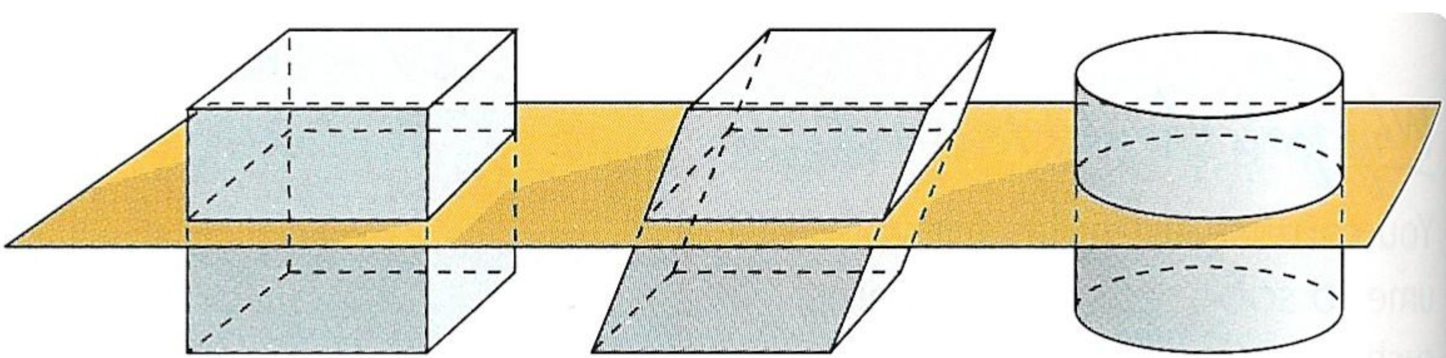
# Volume of a Prism and a Cylinder

Cavalieri's Principle

If two solids have the same height and the same cross-sectional area at every level, then they have the same volume



*Because each stack has the same number of pennies, it follows that each stack has the same volume.*



# Volume of a Prism

- The Volume, V, of a prism is

  $V = Bh$

  where B is the area of a base and h is the height

# Volume of a Cylinder

- The volume, V, of a cylinder is V=Bh or

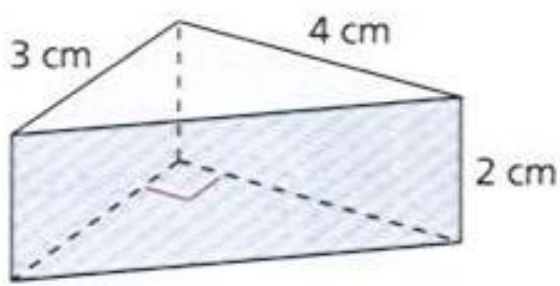
  $V = \pi r^2 h$

  where B is the area of a base, h is the height and r is the radius of a base

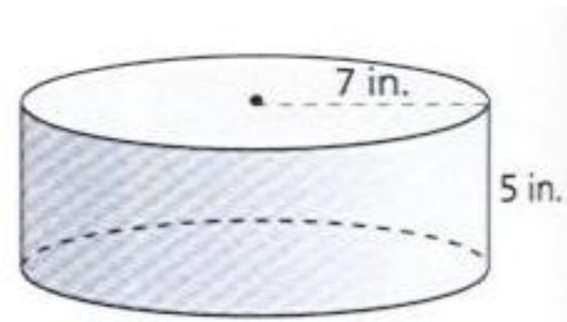
# Example 2
# Finding Volumes

- Find the volume of the right prism and the right cylinder

a.




b.



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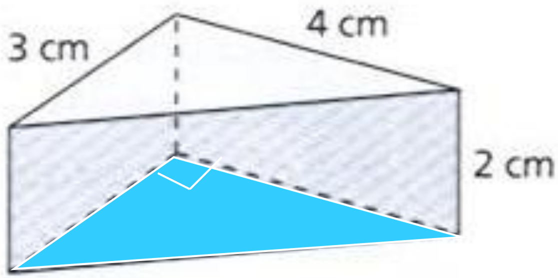



Area of Base

$B = \frac{1}{2}(3)(4)=6$

Height = 2

$V = Bh$

$V = 6(2)$

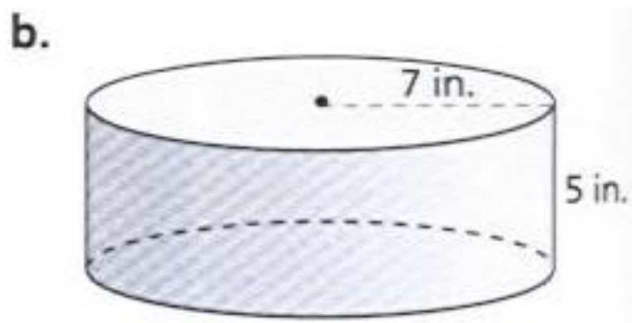
$V = 12\ cm^3$

3 cm

4 cm

# Example 2
# Finding Volumes

- Find the volume of the right prism and the right cylinder




Area of Base

$B = \pi(7)^2 = 49\pi$

Height = 5

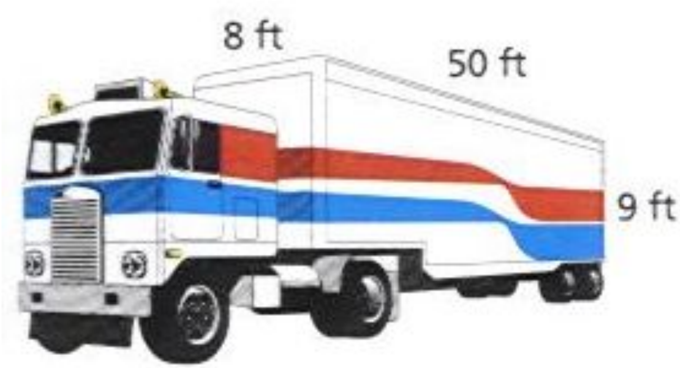
$V = Bh$

$V = 49\pi(5)$

$V = 245\pi \text{ in}^3$

# Example 3
## Estimating the Cost of Moving

- You are moving from Newark, New Jersey, to Golden, Colorado - a trip of 2000 miles. Your furniture and other belongings will fill half the truck trailer. The moving company estimates that your belongings weigh an average of 6.5 pounds per cubic foot. The company charges $600 to ship 1000 pounds. Estimate the cost of shipping your belongings.



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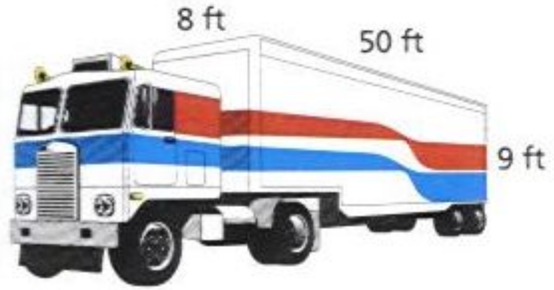


Volume = L x W x H

$V = 50(8)(9)$

$V = 3600 \text{ ft}^3$

$3600 \div 2 = 1800 \text{ ft}^3$

$1800(6.5) = 11{,}700$ pounds

$11{,}700 \div 1000 = 11.7$

$11.7(600) = \$7020$