Karnaugh maps.

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- Minimization of Boolean functions
- Karnaugh maps
- Circuits
- Minimization of circuits

Karnaugh maps

We will describe a procedure simplifying sum-of-products expansions.
The goal of this procedure is to produce Boolean sums of Boolean products that represent a Boolean function with the fewest products of literals such that these products contain the fewest literals possible among all sums of products that represent a Boolean function.
Finding such a sum of products is called **minimization** of the Boolean function.

Karnaugh maps

The procedure we will introduce, known as Karnaugh maps (or K-maps), was designed in the 1950s.

Karnaugh maps

To reduce the number of terms in a Boolean expression it is necessary to find terms to combine.
There is a graphical method, called a Karnaugh map or K-map, for finding terms to combine for Boolean functions involving a relatively small number of variables.
The method we will describe was introduced by Maurice Karnaugh in 1953.
His method is based on earlier work by E. W. Veitch.
(This method is usually applied only when the function involves six or fewer variables.)

Karnaugh maps

MAURICE KARNAUGH (BORN 1924)

Maurice Karnaugh, born in New York City, received his B.S. from the City College of New York and his M.S. and Ph.D. from Yale University.

Karnaugh maps

He was a member of the technical staff at Bell Laboratories from 1952 until 1966 and Manager of Research and Development at the Federal Systems Division of AT&T from 1966 to 1970.

Karnaugh maps

In 1970 he joined IBM as a member of the research staff.

Karnaugh maps

Karnaugh has made fundamental contributions to the application of digital techniques in both computing and telecommunications.

His current interests include knowledge-based systems in computers and heuristic search methods.

Karnaugh maps

K-maps give us a visual method for simplifying sum-of-products expansions; they are not suited for mechanizing this process.

We will first illustrate how K-maps are used to simplify expansions of Boolean functions in two variables.

We will continue by showing how K-maps can be used to minimize Boolean functions in three variables and then in four variables.

Karnaugh maps in two variables

- There are four possible minterms in the sum-of-products expansion of a Boolean function in the two variables x and y.

A K-map for a Boolean function in these two variables consists of four cells, where a 1 is placed in the cell representing a minterm if this minterm is present in the expansion.

Karnaugh maps in two variables

A **literal** is a Boolean variable x or its complement $\bar{x}$.

Hence, a minterm is a product of n literals, with one literal for each variable.

Karnaugh maps in two variables

The four cells and the terms that they represent are shown in the figure.

Karnaugh maps in two variables

- Cells are said to be **adjacent** if the minterms that they represent differ in exactly one literal.

For instance, the cell representing $\bar{x}y$ is adjacent to the cells representing xy and $\bar{x}\,\bar{y}$.

Karnaugh maps in two variables

<u>Example 1</u>

Find the K-map for $xy + \bar{x}y$.

Karnaugh maps in two variables

Example 2

Find the K-map for $x\bar{y} + \bar{x}y$.

Karnaugh maps in two variables

<u>Example 3</u>

Find the K-map for $x\bar{y} + \bar{x}y + \bar{x}\bar{y}$.

Karnaugh maps in two variables

- We can identify minterms that can be combined from the K-map.

Whenever there are 1s in two adjacent cells in the K-map, the minterms represented by these cells can be combined into a product involving just one of the variables.

Karnaugh maps in two variables

- Moreover, if 1s are in all four cells, the four minterms can be combined into one term, namely, the Boolean expression 1 that involves none of the variables.

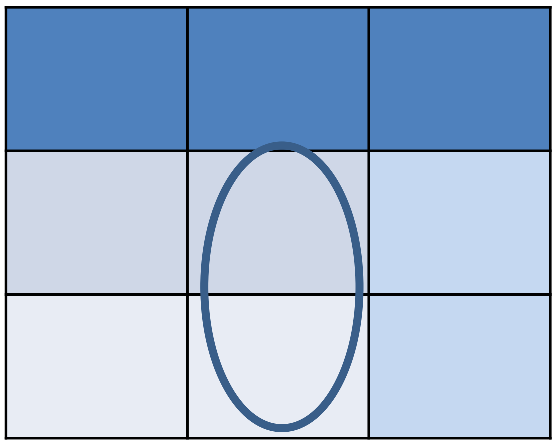
We circle blocks of cells in the K-map that represent minterms that can be combined and then find the corresponding sum of products.

The goal is to identify the largest possible blocks, and to cover all the 1s with the fewest blocks using the largest blocks first and always using the largest possible blocks.

Karnaugh maps in two variables

<u>Example 4</u>

Simplify the sum-of-products expansion $xy + \bar{x}y$.

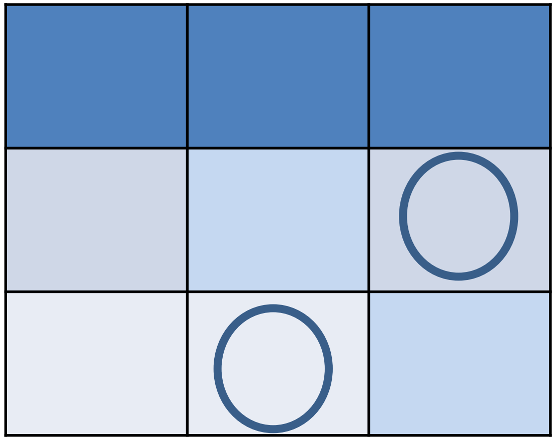


$$xy + \bar{x}y =$$
$$(x + \bar{x})y =$$
$$y$$

Karnaugh maps in two variables

Example 5

Simplify the sum-of-products expansion $x\bar{y} + \bar{x}y$.



$$x\bar{y} + \bar{x}y$$

Karnaugh maps in two variables

<u>Example 6</u>

Simplify the sum-of-products expansion $x\bar{y} + \bar{x}y + \bar{x}\bar{y}$.

		1

Karnaugh maps in two variables

<u>Example 6</u>

Simplify the sum-of-products expansion $x\bar{y} + \bar{x}y + \bar{x}\bar{y}$.

		1

Karnaugh maps in two variables

<u>Example 6</u>

Simplify the sum-of-products expansion $x\bar{y} + \bar{x}y + \bar{x}\bar{y}$.

		1

Karnaugh maps in two variables

-

	y	$\bar{y}$
x		1
$\bar{x}$	1	1

		1

Karnaugh maps in two variables

Example 6

Simplify the sum-of-products expansion $x\bar{y} + \bar{x}y + \bar{x}\bar{y}$.

		1

$$
\begin{aligned}
&x\bar{y} + \bar{x}y + \bar{x}\bar{y} = \\
&(x\bar{y} + \bar{\boldsymbol{x}}\,\bar{\boldsymbol{y}}) + (\bar{x}y + \bar{\boldsymbol{x}}\,\bar{\boldsymbol{y}}) = \\
&(x + \bar{x})\bar{y} + \bar{x}(y + \bar{y}) = \\
&\bar{x} + \bar{y}
\end{aligned}
$$

Karnaugh maps in three variables

A K-map in three variables is a rectangle divided into eight cells.

Karnaugh maps in three variables

Cells are said to be **adjacent** if the minterms that they represent differ in exactly one literal.

The eight cells and the terms that they represent are shown in the figure.

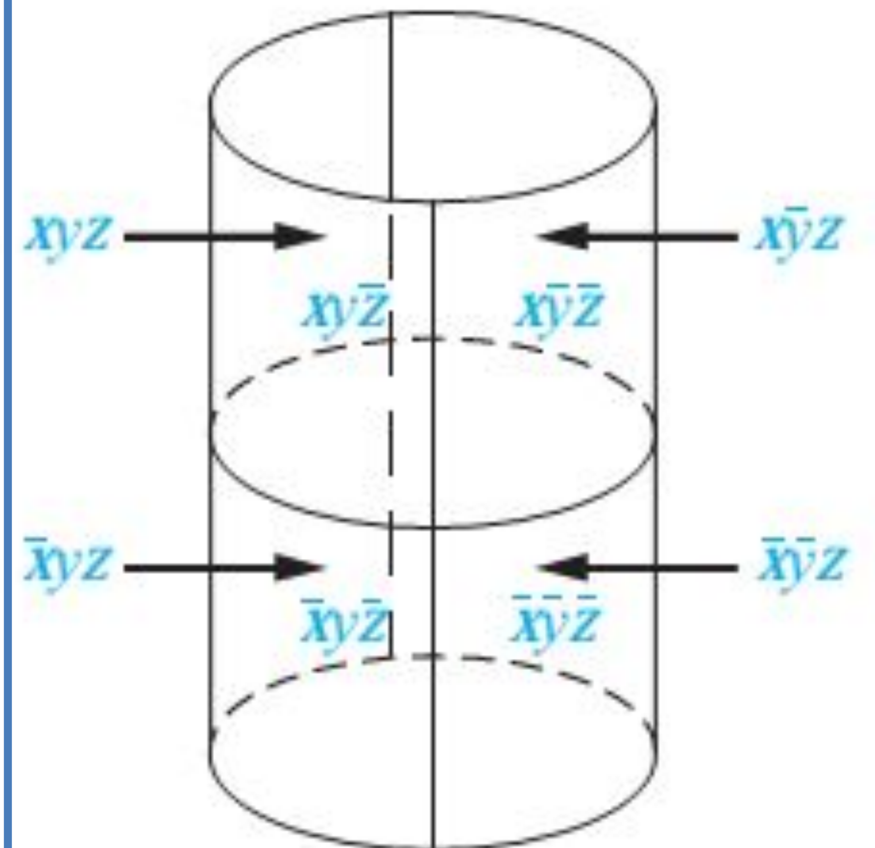
Karnaugh maps in three variables

- For instance, the cell representing $\bar{x}yz$ is adjacent to the cells representing xyz and $\bar{x}\bar{y}z$.

Karnaugh maps in three variables

This K-map can be thought of as lying on a cylinder, as shown in the figure.

On the cylinder, two cells have a common border if and only if they are adjacent.



Karnaugh maps in three variables

- To simplify a sum-of-products expansion in three variables, we use the K-map to identify blocks of minterms that can be combined.

Blocks of two adjacent cells represent pairs of minterms that can be combined into a product of two literals; 2×2 and 4×1 blocks of cells represent minterms that can be combined into a single literal; and the block of all eight cells represents a product of no literals, namely, the function 1.

Karnaugh maps in three variables

- The goal is to identify the largest possible blocks in the map and cover all the 1s in the map with the least number of blocks, using the largest blocks first.

Karnaugh maps in three variables

- Note that there may be more than one way to cover all the 1s using the least number of blocks.

Karnaugh maps in three variables

<u>Example 7</u>

Find the K-map for $x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}$ and simplify the sum-of-products expansion.

$x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z} =$

$(x + \bar{x})\bar{y}\bar{z} =$

$\bar{y}\bar{z}$

Karnaugh maps in three variables

<u>Example 8</u>

Find the K-map for $\bar{x}yz + \bar{x}\bar{y}z$ and simplify the sum-of-products expansion.

$$\bar{x}yz + \bar{x}\bar{y}z =$$
$$\bar{x}z(y + \bar{y}) =$$
$$\bar{x}z$$

Karnaugh maps in three variables

Example 9

Find the K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$ and simplify the sum-of-products expansion.

$$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} =$$
$$(xy\bar{z} + x\bar{y}\bar{z}) + (\bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}) =$$
$$x\bar{z}(y + \bar{y}) + \bar{x}\bar{z}(y + \bar{y}) =$$
$$x\bar{z} + \bar{x}\bar{z} =$$
$$(x + \bar{x})\bar{z} =$$
$$\bar{z}$$

Karnaugh maps in three variables

<u>Example 10</u>

Find the K-map for $\bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$ and simplify the sum-of-products expansion.

$$\bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z =$$
$$\bar{x}(yz + y\bar{z} + \bar{y}\bar{z} + \bar{y}z) =$$
$$\bar{x}((yz + y\bar{z}) + (\bar{y}\bar{z} + \bar{y}z)) =$$
$$\bar{x}(y(z + \bar{z}) + \bar{y}(\bar{z} + z)) =$$
$$\bar{x}(y + \bar{y}) =$$
$$\bar{x}$$

Karnaugh maps in three variables

<u>Example 10</u>

Find the K-map for

$xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$

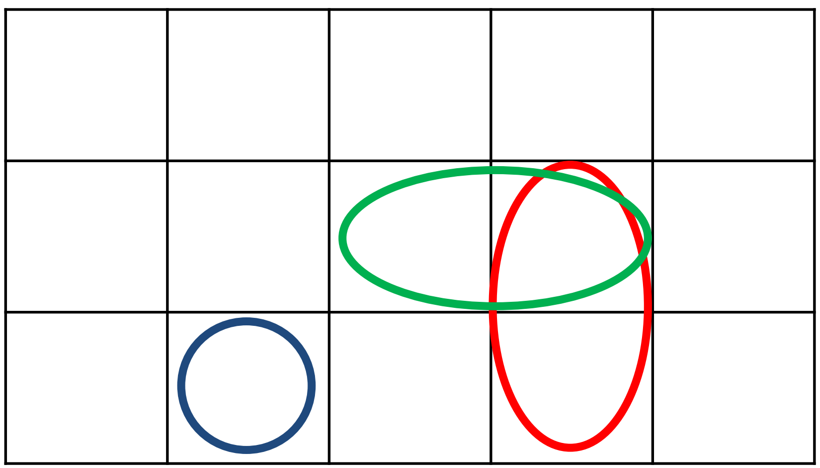
and simplify the sum-of-products expansion.

$(xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z) +$
$(\bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z) =$
$x + \bar{x} =$
1

Karnaugh maps in three variables

<u>Example 11</u>

Find the K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$ and simplify the sum-of-products expansion.



Karnaugh maps in three variables

<u>Example 11</u>

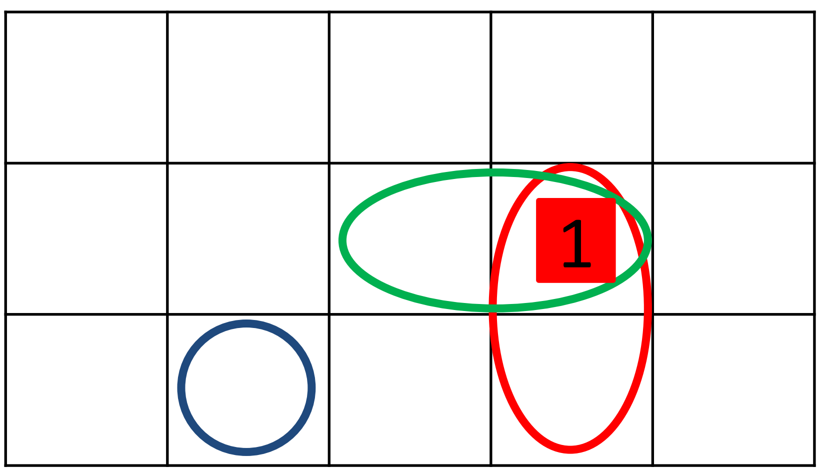
Find the K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$ and simplify the sum-of-products expansion.

			1	

Karnaugh maps in three variables

Example 11

Find the K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$ and simplify the sum-of-products expansion.



Karnaugh maps in three variables

<u>Example 11</u>

Find the K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$ and simplify the sum-of-products expansion.

			1	

Example 11

Find the K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$ and simplify the sum-of-products expansion.

			1	

$$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z} =$$
$$(xy\bar{z} + \boldsymbol{x\bar{y}\bar{z}}) + \bar{x}yz + (\bar{x}\bar{y}\bar{z} + \boldsymbol{x\bar{y}\bar{z}})$$
$$= x\bar{z} + \bar{x}yz + \bar{y}\bar{z}$$

Karnaugh maps in three variables

<u>Example 12</u>

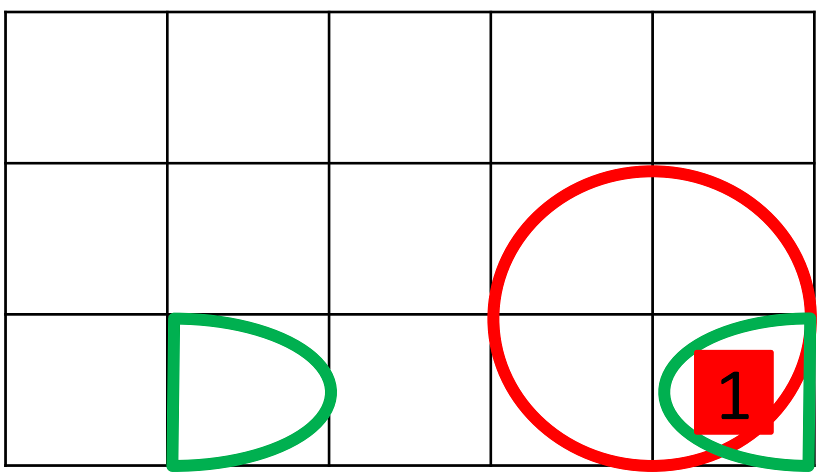
Find the K-map for $x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$ and simplify the sum-of-products expansion.

				1

Karnaugh maps in three variables

<u>Example 12</u>

Find the K-map for $x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$ and simplify the sum-of-products expansion.



Karnaugh maps in three variables

<u>Example 12</u>

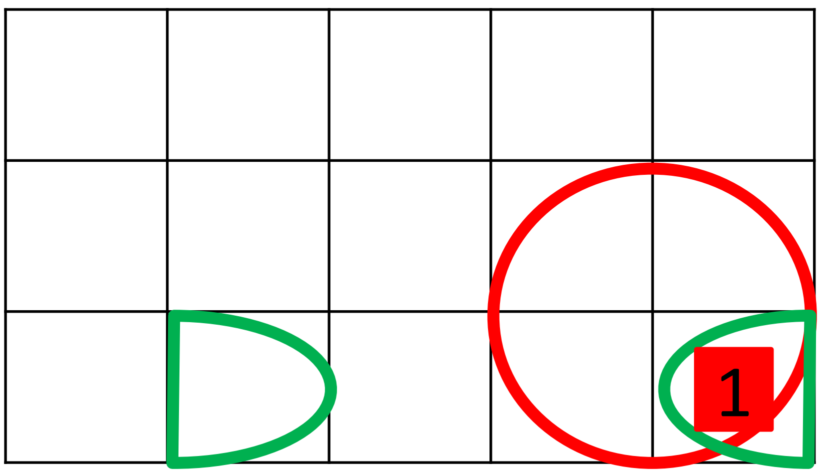
Find the K-map for $x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$ and simplify the sum-of-products expansion.

				1

Karnaugh maps in three variables

<u>Example 12</u>

Find the K-map for $x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$ and simplify the sum-of-products expansion.



$x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z =$
$(x\bar{y}\bar{z} + x\bar{y}z + \boldsymbol{\bar{x}\,\bar{y}z} + \bar{x}\bar{y}\bar{z}) +$
$+(\boldsymbol{\bar{x}\,\bar{y}z} + \bar{x}yz) =$
$\bar{y} + \bar{x}z$

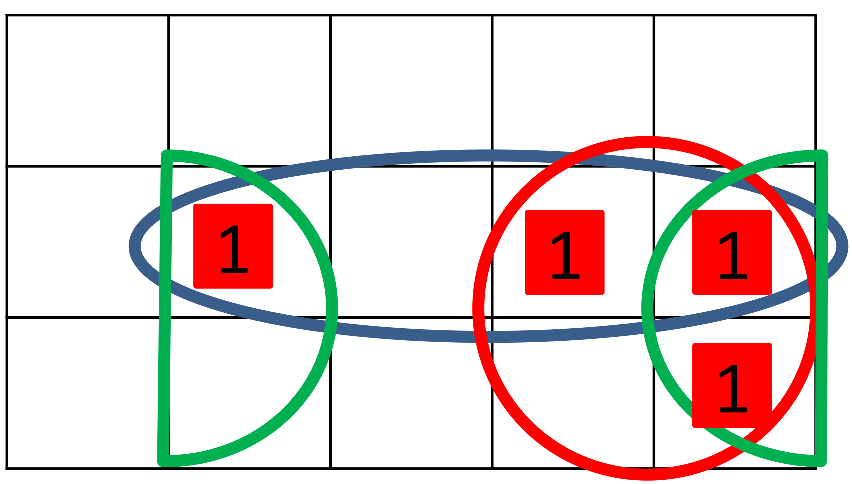
Karnaugh maps in three variables

<u>Example 13</u>

Find the K-map for

$$xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$$

and simplify the sum-of-products expansion.



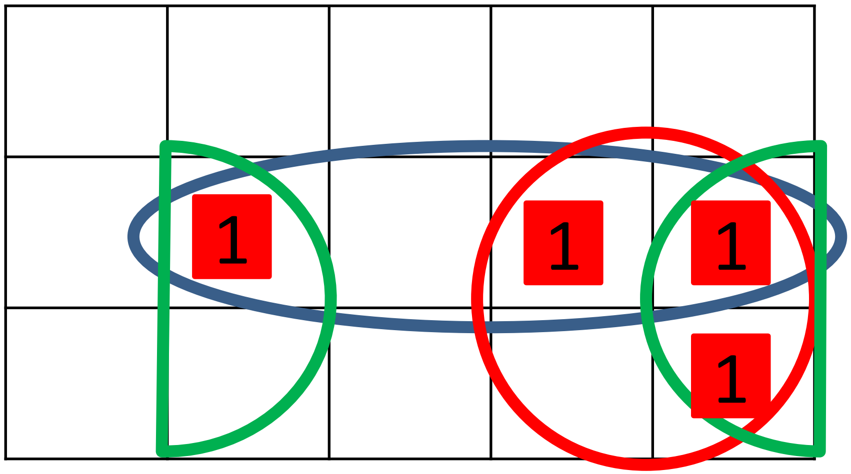
Karnaugh maps in three variables

<u>Example 13</u>

Find the K-map for

$$xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$$

and simplify the sum-of-products expansion.



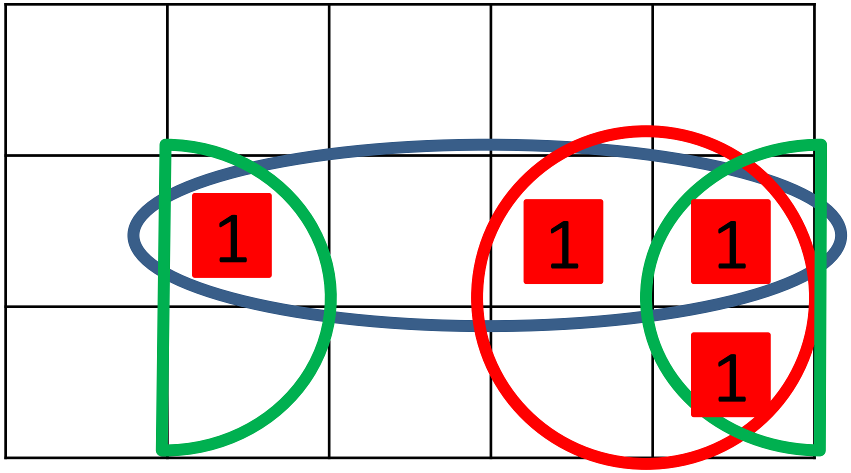
Karnaugh maps in three variables

<u>Example 13</u>

Find the K-map for

$$xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$$

and simplify the sum-of-products expansion.



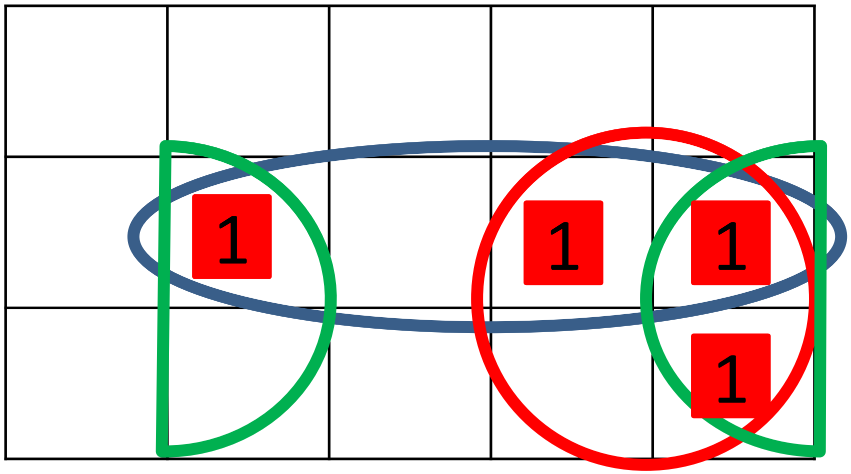
Karnaugh maps in three variables

Example 13

Find the K-map for

$xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$

and simplify the sum-of-products expansion.

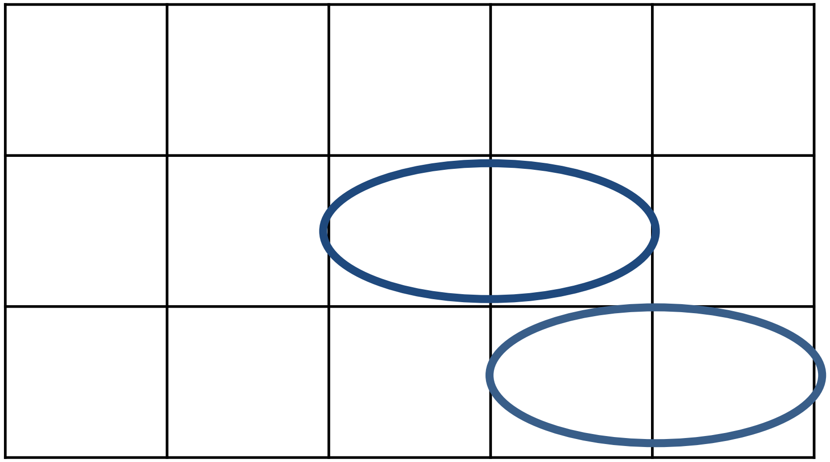


$x + \bar{y} + z$

Karnaugh maps in three variables

<u>Example 14</u>

Find the K-map for $xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$ and simplify the sum-of-products expansion.



$x\bar{z} + \bar{x}\bar{y}$

Karnaugh maps in four variables

The sixteen cells and the terms that they represent are shown in the figure.

Karnaugh maps in four variables

Cells are said to be **adjacent** if the minterms that they represent differ in exactly one literal.

Karnaugh maps in four variables

- For instance, the cell representing $w\bar{x}yz$ is adjacent to the cells representing $wxyz$ and $w\bar{x}\bar{y}z$.

Karnaugh maps in four variables

The K-map of a sum-of-products expansion in four variables can be thought of as lying on a torus, so that adjacent cells have a common boundary.

Karnaugh maps in four variables

- The simplification of a sum-of-products expansion in four variables is carried out by identifying those blocks of $2, 4, 8$, or 16 cells that represent minterms that can be combined.

Each cell representing a minterm must either be used to form a product using fewer literals, or be included in the expansion.

Karnaugh maps in four variables

- As is the case in K-maps in two and three variables, the goal is to identify the largest blocks of 1s in the map and to cover all the 1s using the fewest blocks needed, using the largest blocks first.

The largest possible blocks are always used.

Example 15 Simplify the sum-of-products expansion $w\bar{x}yz + wx\bar{y}\bar{z}$

$$w\bar{x}yz + w\bar{x}\bar{y}z =$$
$$w\bar{x}z(y + \bar{y}) =$$
$$w\bar{x}z$$

Example 16 Simplify the sum-of-products expansion $\bar{w}\bar{x}\bar{y}z + \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}yz$.

$$\bar{w}\bar{x}\bar{y}z + \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}yz =$$
$$\bar{w}\bar{x}(\bar{y}z + \bar{y}\bar{z} + y\bar{z} + yz) =$$
$$\bar{w}\bar{x}(\bar{y}(z + \bar{z}) + y(\bar{z} + z)) =$$
$$\bar{w}\bar{x}(\bar{y} + y) =$$
$$\bar{w}\bar{x}$$

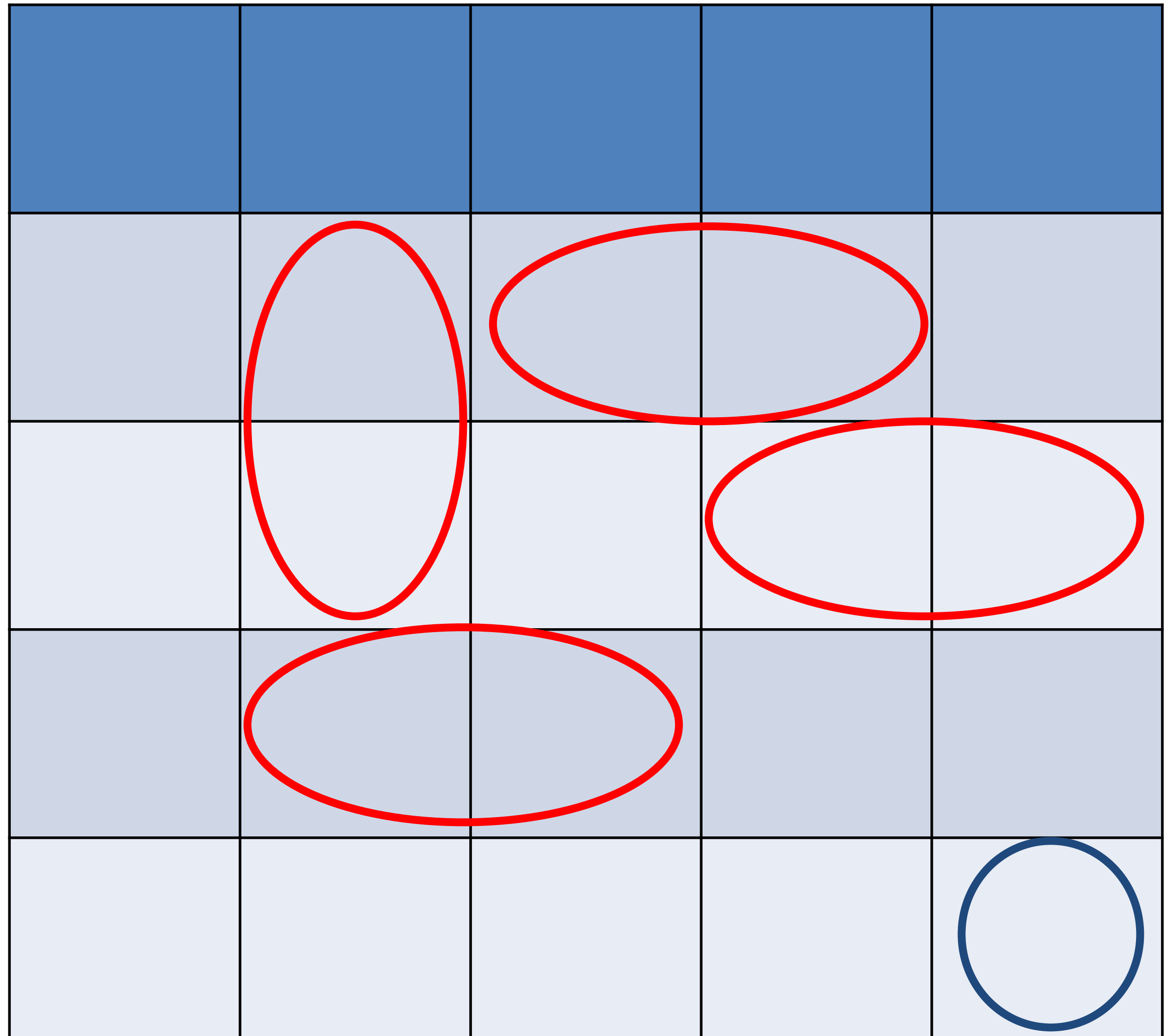
Example 17 Simplify the sum-of-products expansion $wxyz + wx\bar{y}z + \bar{w}xyz + \bar{w}x\bar{y}z$

$$wxyz + wx\bar{y}z + \bar{w}xyz + \bar{w}x\bar{y}z =$$
$$xz(wy + w\bar{y} + \bar{w}y + \bar{w}\bar{y}) =$$
$$xz(w(y + \bar{y}) + \bar{w}(y + \bar{y})) =$$
$$xz(w + \bar{w}) =$$
$$xz$$

Example 18 Simplify the sum-of-products expansion $wxy\bar{z} + wx\bar{y}\bar{z} + w\bar{x}\,y\bar{z} + w\bar{x}\bar{y}\bar{z} +$
$\bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}xy\bar{z} + \bar{w}x\bar{y}\bar{z}$

$$(wxy\bar{z} + w\bar{x}\,y\bar{z} + \bar{w}\bar{x}y\bar{z} + \bar{w}xy\bar{z}) + (wx\bar{y}\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}x\bar{y}\bar{z})$$
$$= y\bar{z} + \bar{y}\bar{z} = (y + \bar{y})\bar{z} = \bar{z}$$

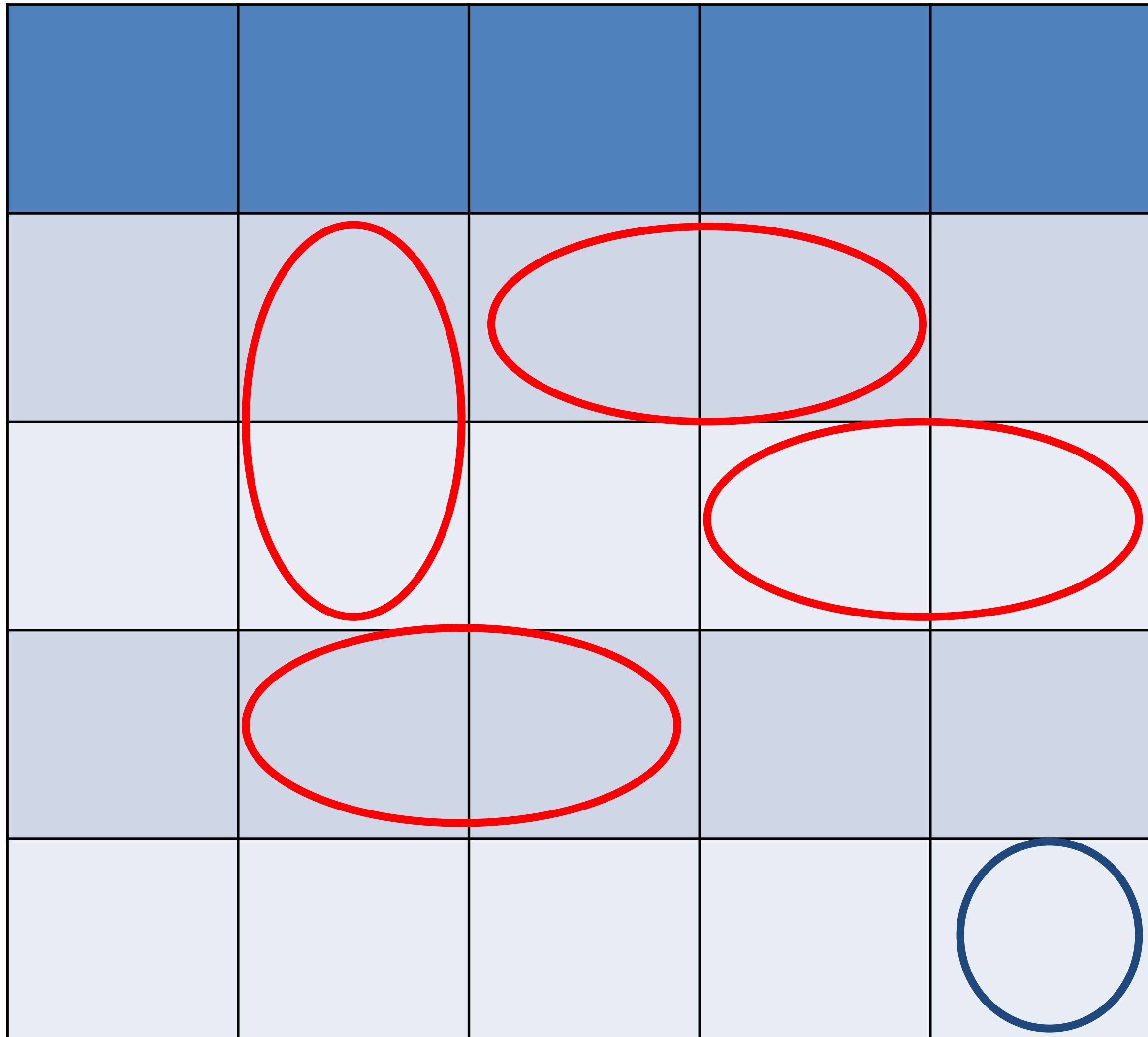
<u>Example 19</u> Simplify the sum-of-products expansion



Example 19 Simplify the sum-of-products expansion

Solution:

$wyz + wx\bar{z}$
$+ w\bar{x}\bar{y} + \bar{w}\bar{x}y$
$+ \bar{w}x\bar{y}z$



Example 20 Simplify the sum-of-products expansion

		1	1	
			1	

Example 20 Simplify the sum-of-products expansion

		1	1	
			1	

Example 20 Simplify the sum-of-products expansion

		1	1	
			1	

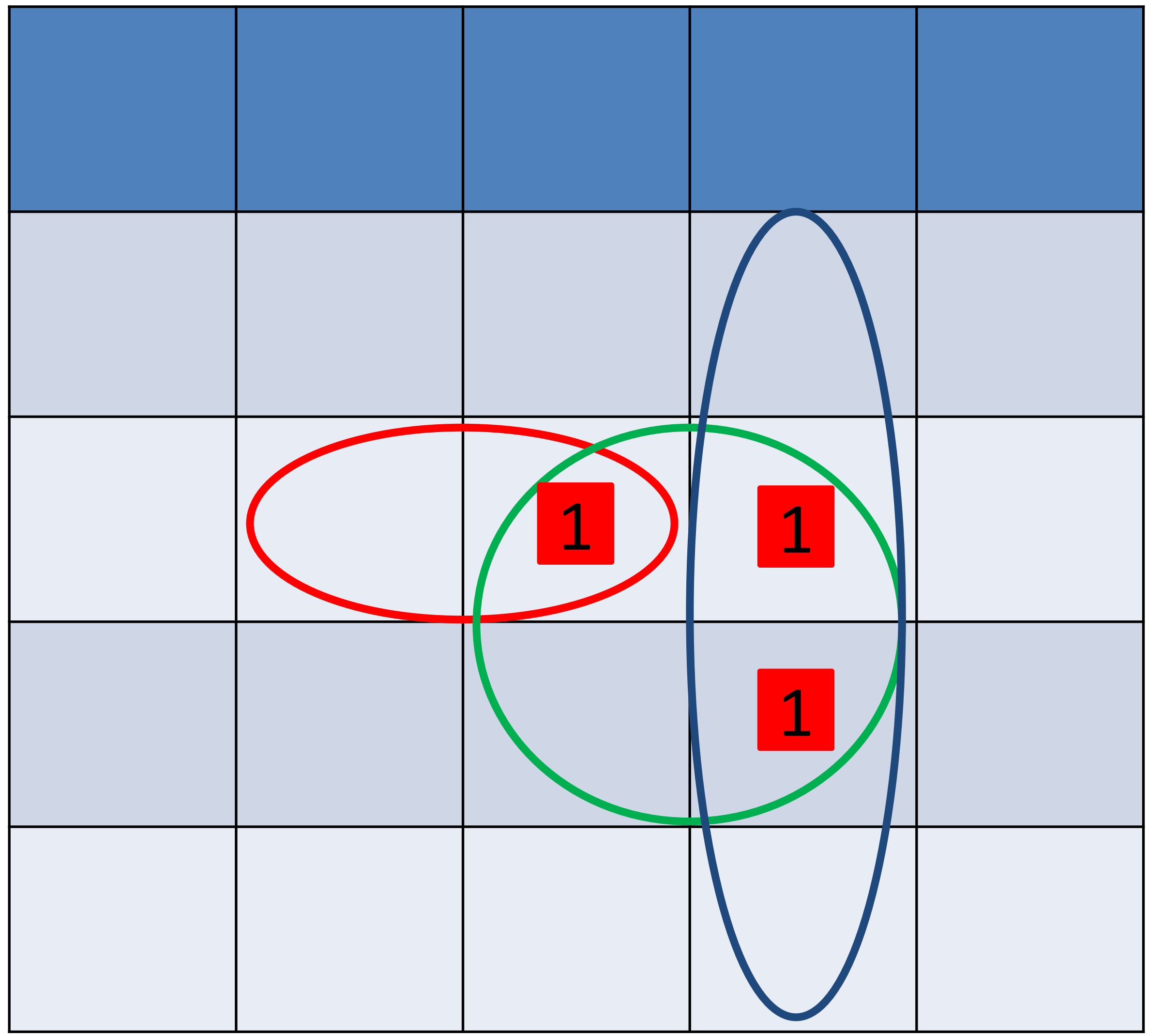
Example 20 Simplify the sum-of-products expansion

		1	1	
			1	

<u>Example 20</u> Simplify the sum-of-products expansion

<u>Solution:</u>

$w\bar{x}y + \bar{x}\bar{z} + \bar{y}\bar{z}$



<u>Example 21</u> Simplify the sum-of-products expansion

		1		
		1	1	

Example 21 Simplify the sum-of-products expansion

		1		
		1	1	

<u>Example 21</u> Simplify the sum-of-products expansion

		1		
		1	1	

Example 21 Simplify the sum-of-products expansion

		1		
		1	1	

Example 21 Simplify the sum-of-products expansion

Solution:

$w\bar{x}y + \bar{z} + \bar{w}x$

		1		
		1	1	

Circuits

- Boolean algebra is used to model the circuitry of electronic devices.

Each input and each output of such a device can be thought of as a member of the set $\{0,1\}$.

A computer, or other electronic device, is made up of a number of circuits.

Each circuit can be designed using the rules of Boolean algebra.

Circuits

The basic elements of circuits are called gates.

Each type of gate implements a Boolean operation.

We define several types of gates. Using these gates, we will apply the rules of Boolean algebra to design circuits that perform a variety of tasks.

The circuits that we will study give output that depends only on the input, and not on the current state of the circuit. In other words, these circuits have no memory capabilities.

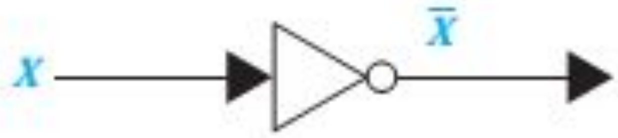
Such circuits are called combinational circuits or gating networks.

Logic gates

We will construct combinational circuits using three types of elements.

The first is an inverter, which accepts the value of one Boolean variable as input and produces the complement of this value as its output.

The symbol used for an inverter is shown in the figure.



Logic gates

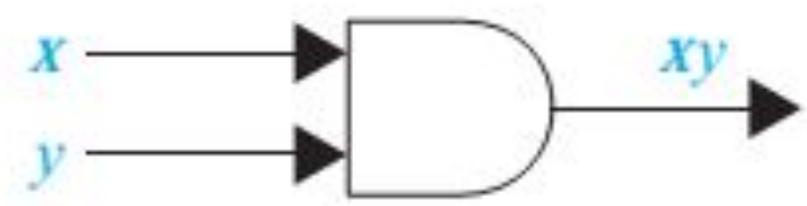
- The next type of element we will use is the OR gate. The inputs to this gate are the values of two or more Boolean variables. The output is the Boolean sum of their values. The symbol used for an OR gate is shown in the figure.

Logic gates

- The third type of element we will use is the AND gate. The inputs to this gate are the values of two or more Boolean variables.

The output is the Boolean product of their values.

The symbol used for an AND gate is shown in the figure.



Circuits

- Combinational circuits can be constructed using a combination of inverters, OR gates, and AND gates.

Circuits

The efficiency of a combinational circuit depends on the number and arrangement of its gates.

The process of designing a combinational circuit begins with the table specifying the output for each combination of input values.

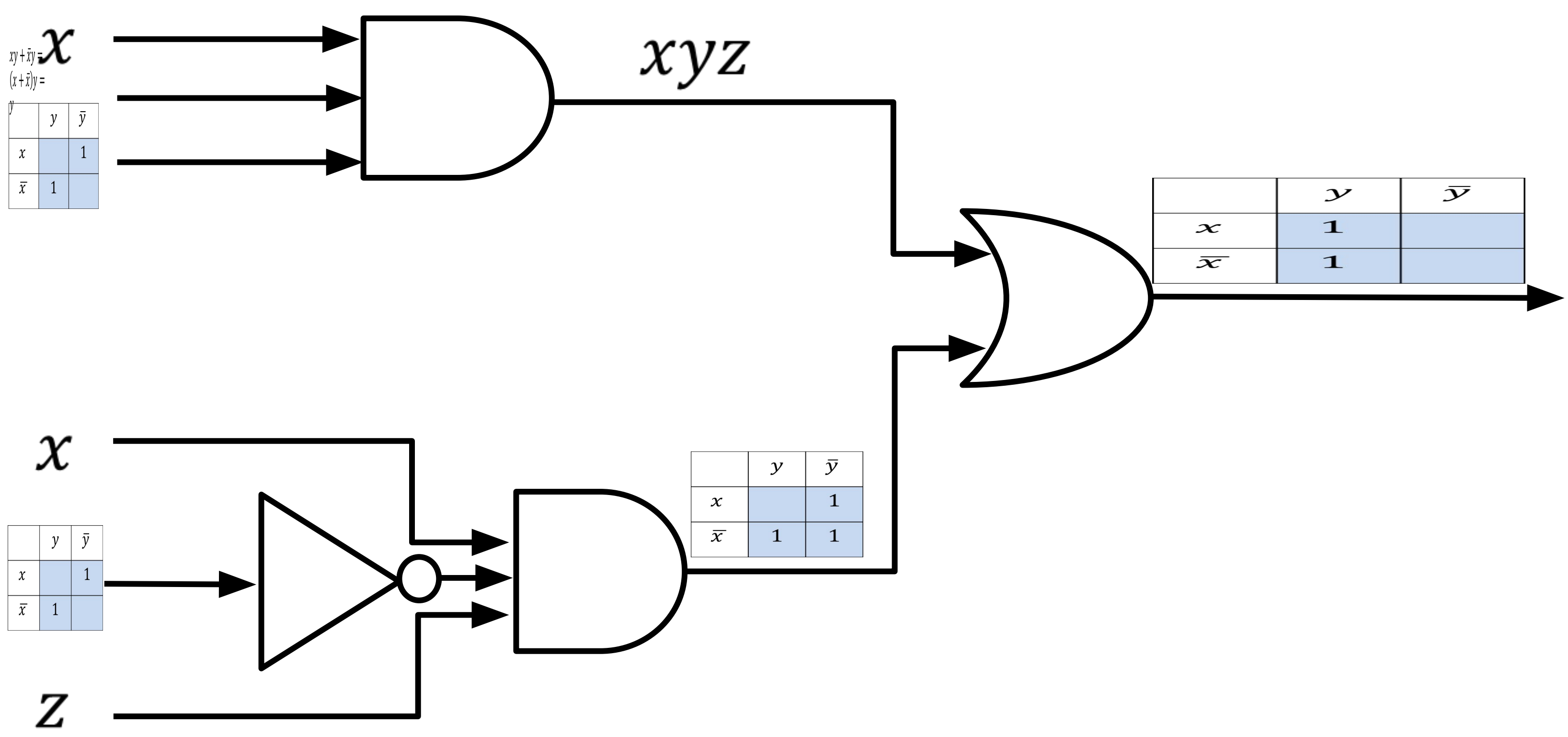
We can always use the sum-of-products expansion of a circuit to find a set of logic gates that will implement this circuit.

Minimization of circuits

- However, the sum-of-products expansion may contain many more terms than are necessary.

Minimizing a Boolean function makes it possible to construct a circuit for this function that uses the fewest gates and fewest inputs to the AND gates and OR gates in the circuit, among all circuits for the Boolean expression we are minimizing.

Use K-maps to find simpler circuits with the same output as the circuit shown.



Use K-maps to find simpler circuits with the same output as the circuit shown.

Let's find the K-map for $xyz + x\bar{y}z$ and simplify the sum-of-products expansion.

	y	$\bar{y}$
x		1
$\bar{x}$	1	1

Use K-maps to find simpler circuits with the same output as the circuit shown.

- $$x\bar{y} + \bar{x}y + \bar{x}\bar{y} =$$

$$\left(x\bar{y} + \bar{x}\bar{y}\right) + \left(\bar{x}y + \bar{x}\bar{y}\right) =$$

$$\left(x + \bar{x}\right)\bar{y} + \bar{x}\left(y + \bar{y}\right) =$$

$$\bar{x} + \bar{y}$$