Combinatorics

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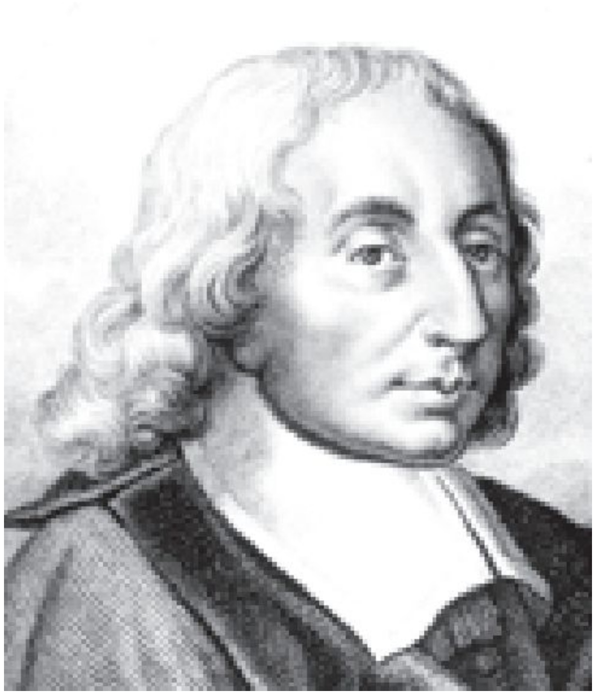
- Pascal’s identity and triangle
- Permutations with repetition
- Combinations with repetition
- Permutations with indistinguishable objects
- Rearrangement theorem

Pascal’s identity and triangle

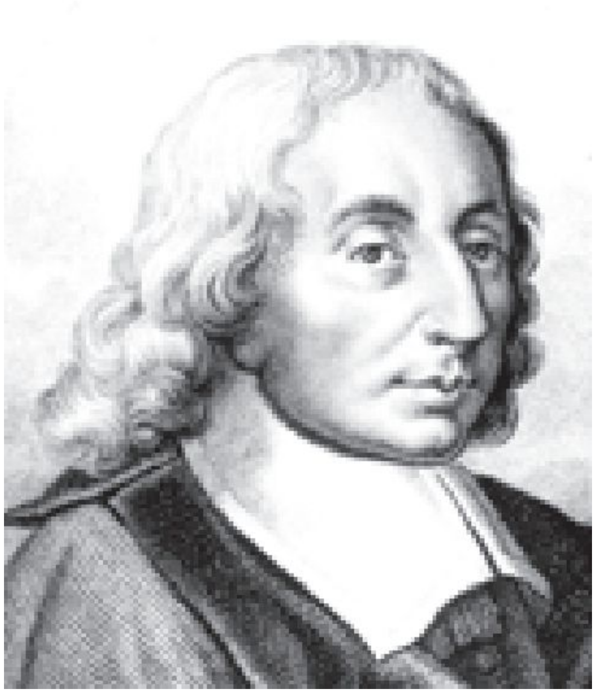
Blaise Pascal exhibited his talents at an early age, although his father, who had made discoveries in analytic geometry, kept mathematics books away from him to encourage other interests.



Blaise Pascal
(1623–1662)

Pascal's identity and triangle

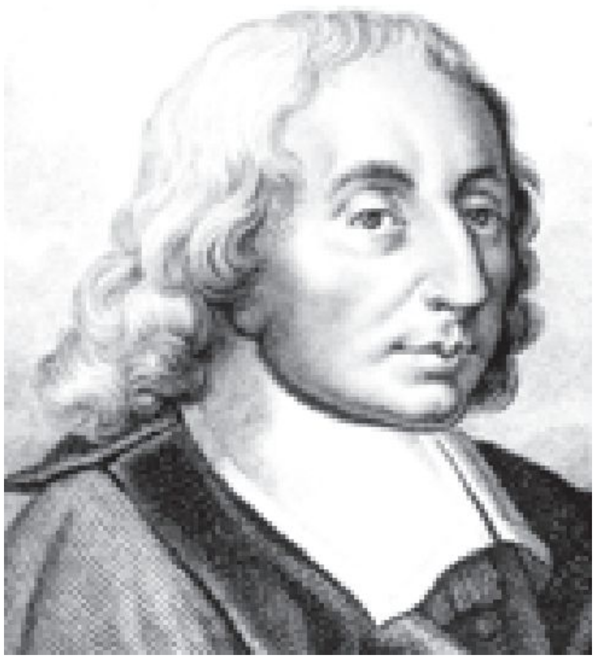
At 16 Pascal discovered an important result concerning conic sections. At 18 he designed a calculating machine, which he built and sold. Pascal, along with Fermat, laid the foundations for the modern theory of probability.



Blaise Pascal
(1623–1662)

Pascal's identity and triangle

In this work, he made new discoveries concerning what is now called Pascal's triangle.
In 1654, Pascal abandoned his mathematical pursuits to devote himself to theology.



Blaise Pascal
(1623–1662)

Pascal's identity and triangle

Theorem 4

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof:

Suppose that T is a set containing $n+1$ elements.

Let $a \in T$, and let $S = T - \{a\}$.

Note that there are $\binom{n+1}{k}$ subsets of T, containing k elements.

A subset of T with k elements

1) **either contains a together with $k-1$ elements of S**,
2) **or contains k elements of S and does not contain a**.

Pascal's identity and triangle

Theorem 4

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof

Because there are $\binom{n}{k-1}$ subsets of $k-1$ elements of S, there are $\binom{n}{k-1}$ **subsets of k elements of T that contain a.**

And there are $\binom{n}{k}$ **subsets of k elements of T that do not contain a**, because there are $\binom{n}{k}$ subsets of k elements of S.

Consequently,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \quad \blacksquare$$

Pascal's identity and triangle

-

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

$$\cdots$$

Pascal's identity and triangle

- The nth row in the triangle consists of the binomial coefficients

$$\binom{n}{k}, k = 0, 1, \ldots, n.$$

Pascal's identity,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

together with the initial conditions

$$\binom{n}{0} = \binom{n}{n} = 1, \text{ for all integers } n,$$

can be used to **recursively** define binomial coefficients.

Pascal's identity and triangle

- 1

  1 1

  1 2 1

  1 3 3 1

  1 4 6 4 1

  1 5 10 10 5 1

  1 6 15 20 **15** **6** 1

  1 7 21 35 35 **21** 7 1

  1 8 28 56 70 56 28 8 1

  ...

Problems

<u>Problem 1</u>

Proof Vandermond's identity :

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Problems

Problem 2

If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

Proof!

Hint:

We use Vandermonde's identity with $m = r = n$.

Permutations with repetition

Example 1

How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution:

By the product rule, because there are 26 upper case English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r. ■

Permutations with repetition

<u>Theorem 1</u>

The number of r-permutations of a set of n objects with repetition allowed is n^r.

<u>Proof</u>:

There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed, because for each choice all n objects are available.

Hence, by the product rule there are n^r r-permutations when repetition is allowed. ■

Combinations with repetition

Example 2

How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills?

Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

Combinations with repetition

Example 2

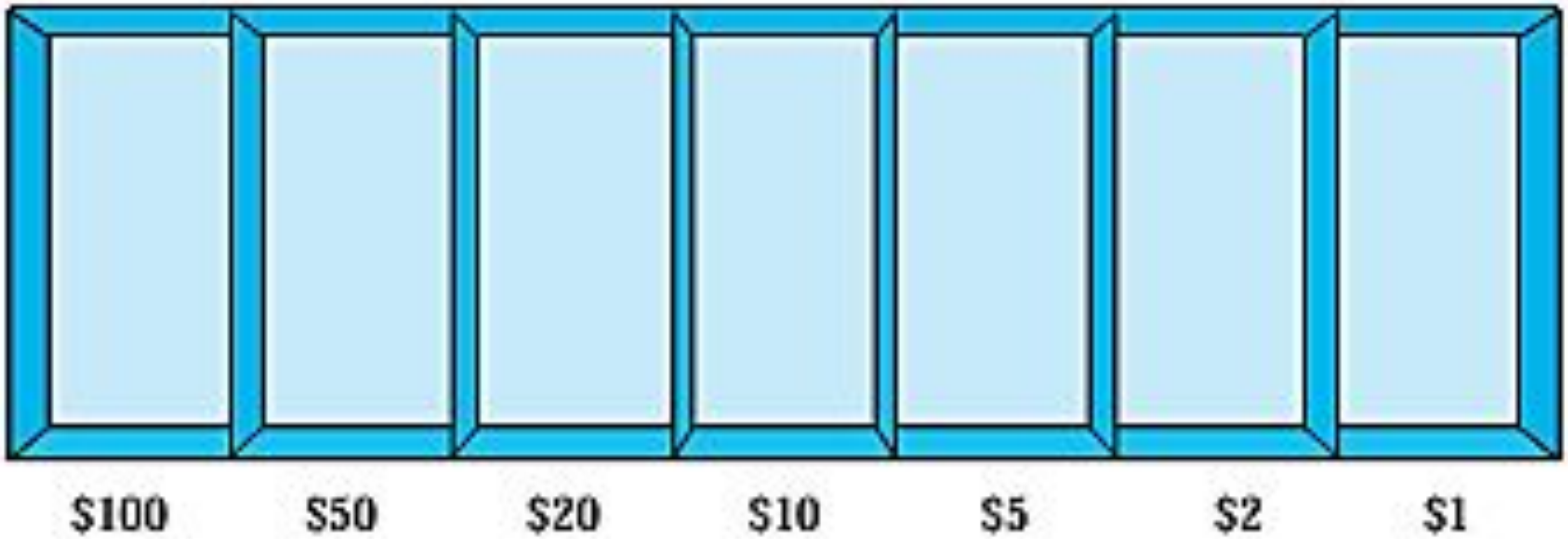
Solution: Because the order in which the bills are selected does not matter and seven different types of bills can be selected as many as five times, this problem involves counting 5-combinations with repetition allowed from a set with seven elements.

Listing all possibilities would be tedious, because there are a large number of solutions. Instead, we will illustrate the use of a technique for counting combinations with repetition allowed.

Solution of example 2:

Suppose that a cash box has seven compartments, one to hold each type of bill, as illustrated in the figure.

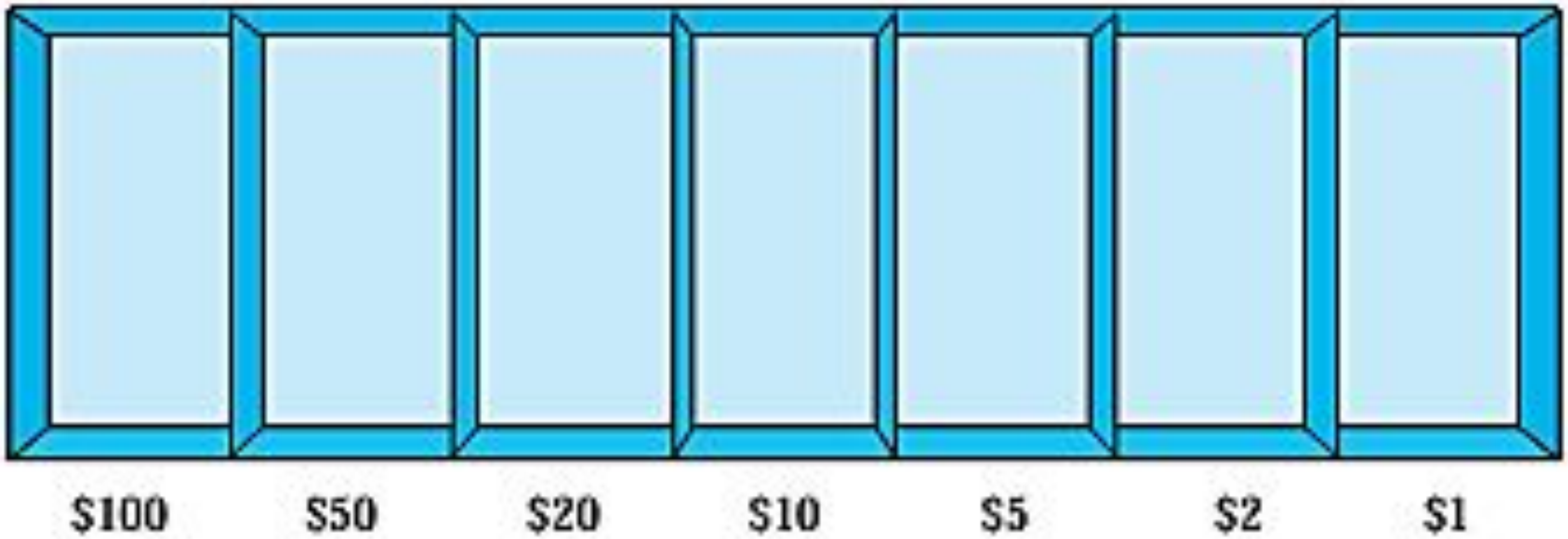
Cash box with seven types of bills:



Solution of example 2:

These compartments are separated by six dividers, as shown in the picture.

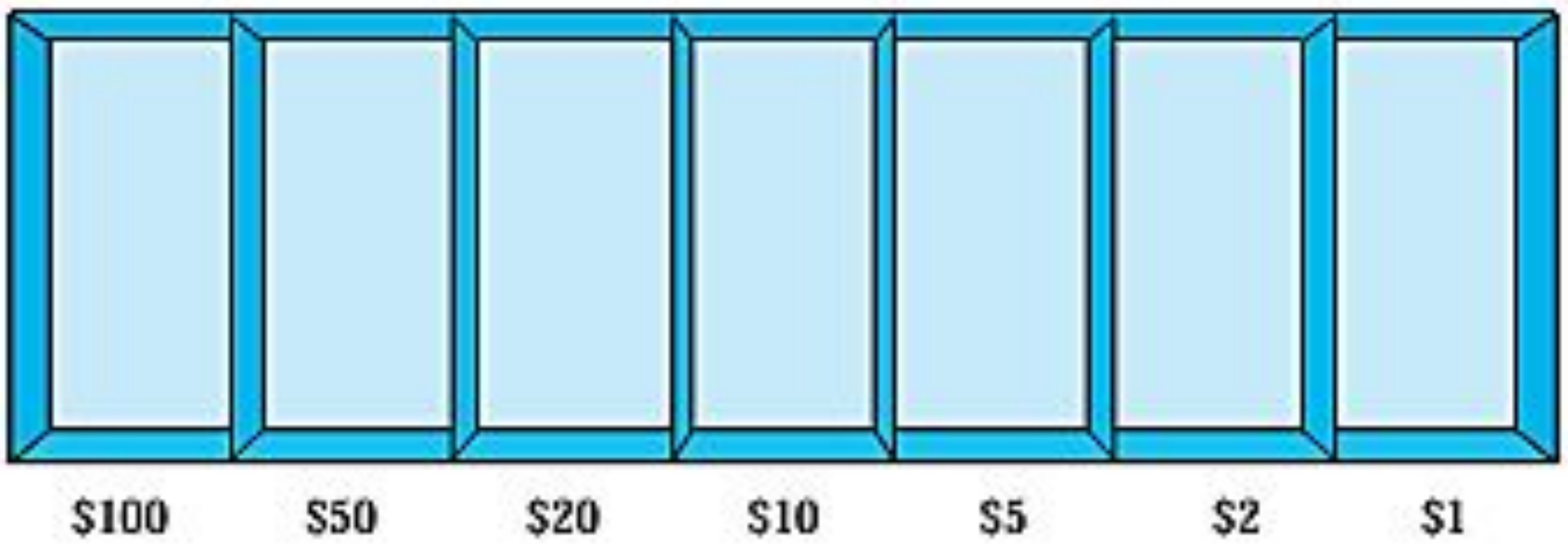
Cash box with seven types of bills:



Solution of example 2:

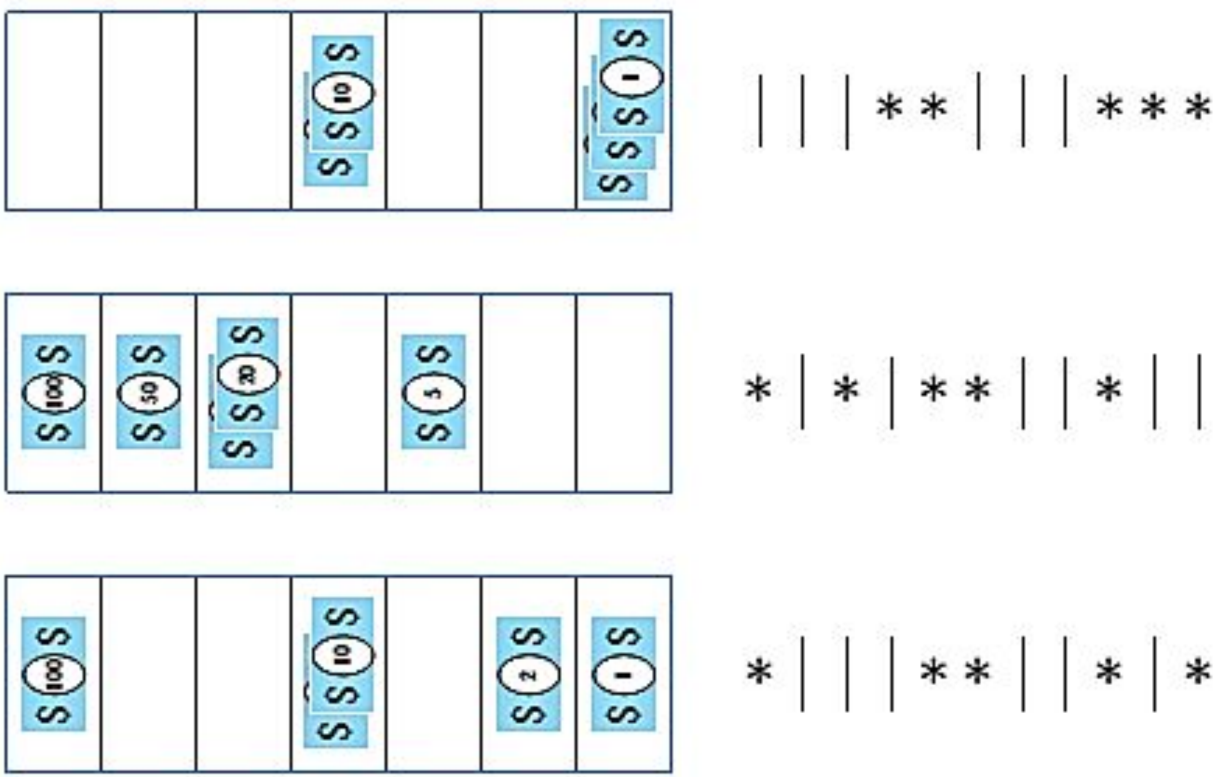
The choice of five bills corresponds to placing five markers in the compartments holding different types of bills.

Cash box with seven types of bills:



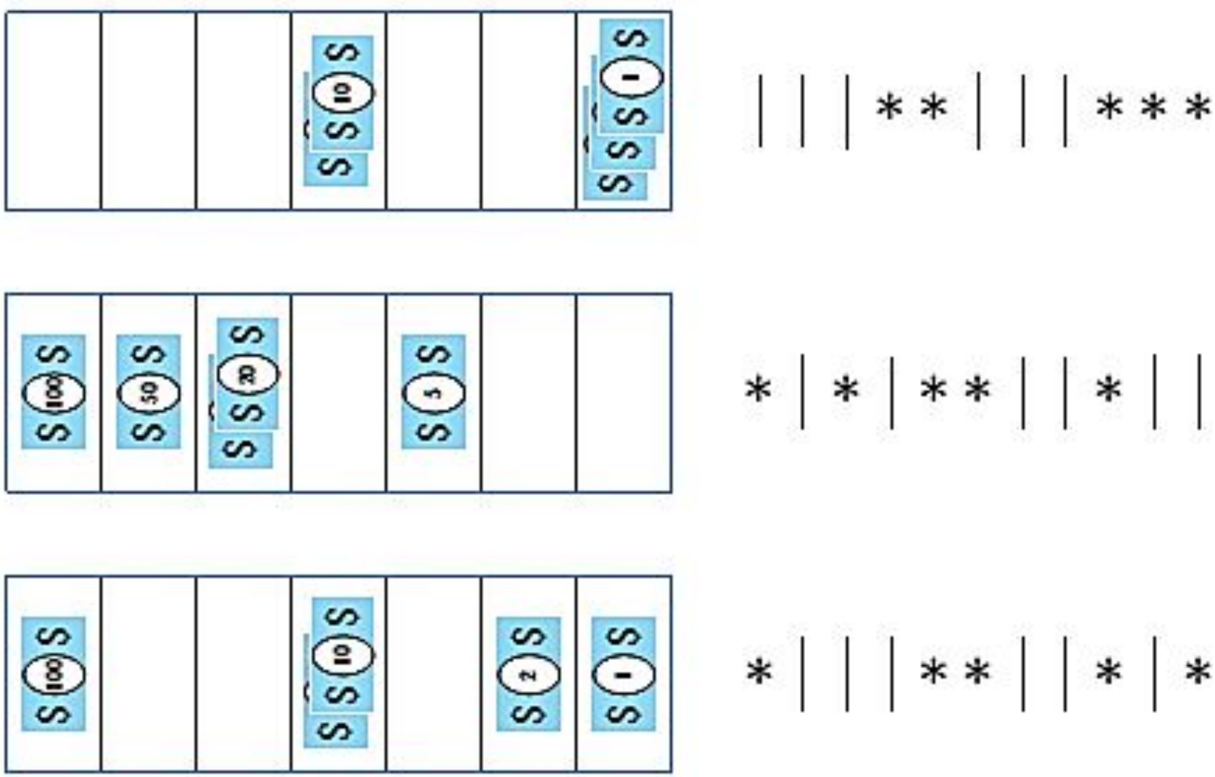
Solution of example 2:

We illustrate this correspondence for three different ways to select five bills, where the six dividers are represented by bars and the five bills by stars.



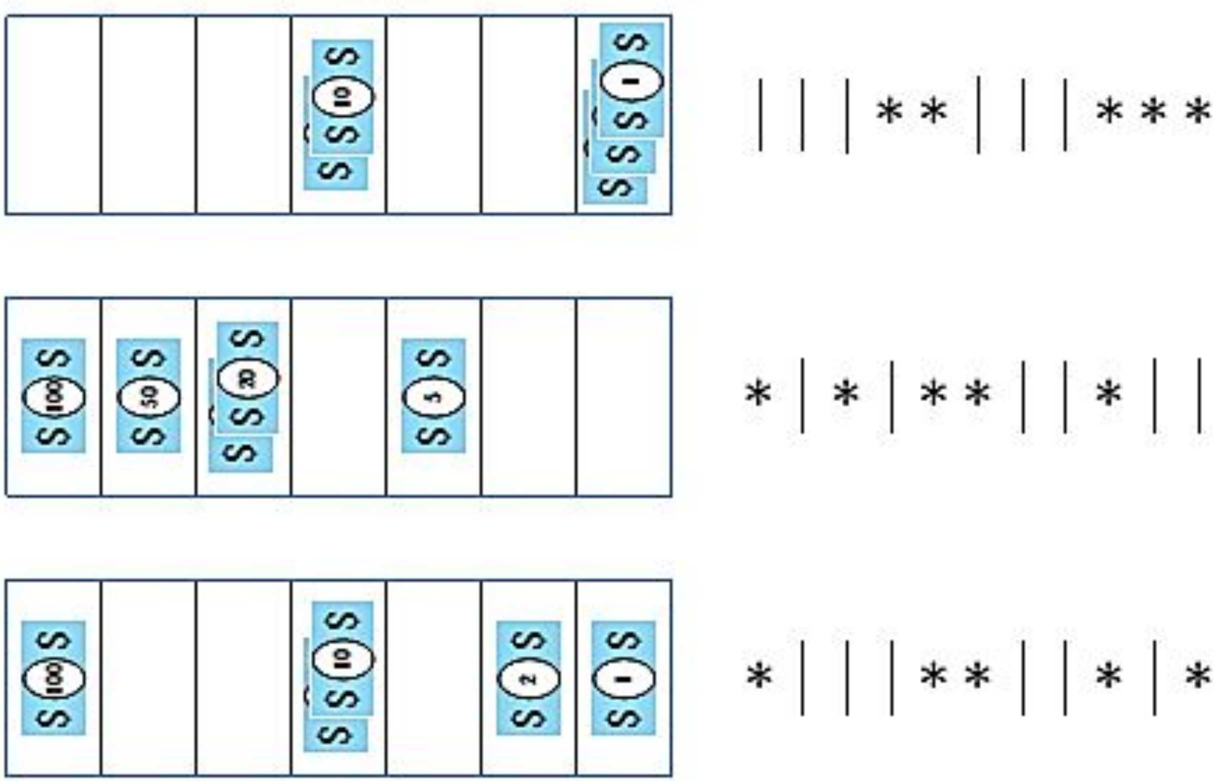
Solution of example 2:

The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row with a total of 11 positions.



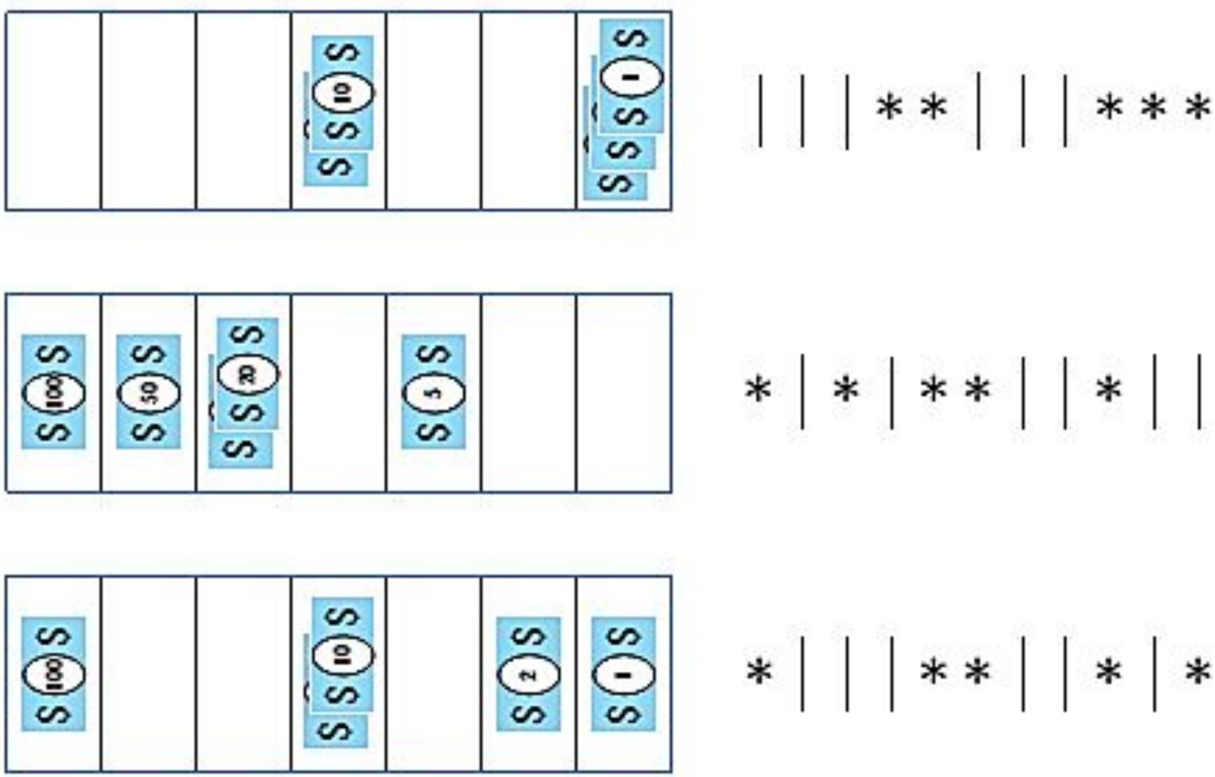
Solution of example 2:

Consequently, the number of ways to select the five bills is the number of ways to select the positions of the five stars from the 11 positions.



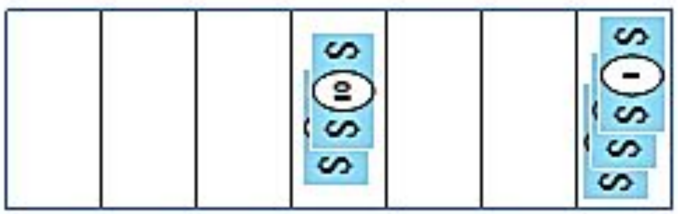
Solution of example 2:

- This corresponds to the number of unordered selections of 5 objects from a set of 11 objects, which can be done in $C(11,5)$ ways.



Solution of example 2:

- Consequently, there are $C(11,5) = \frac{11!}{5!6!}$ =462 ways to choose five bills from the cash box with seven types of bills.



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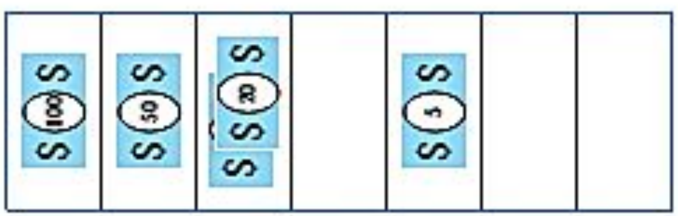

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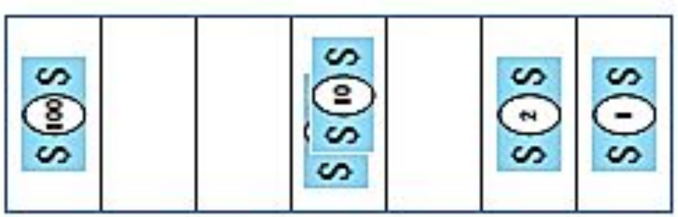

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Combinations with repetition

Theorem 2 There are

$$C(n+r-1, r) = C(n+r-1, n-1\ ) = \frac{(n+r-1)!}{n!(r-1)!}$$

r-combinations from a set with n elements when repetition of elements is allowed.

Proof:

Each r-combination of a set with n elements when repetition is allowed can be represented by a list of $n-1$ bars and r stars.

The $n-1$ bars are used to mark off n different cells, with the ith cell containing a star for each time the ith element of the set occurs in the combination.

Theorem 2 There are

$$C(n+r-1,r) = C(n+r-1,n-1) = \frac{(n+r-1)!}{n!(r-1)!}$$

r-combinations from a set with n elements when repetition of elements is allowed.

Proof:

For instance, a 6-combination of a set with four elements is represented with three bars and six stars.

Here ****|*||***** represents the combination containing exactly two of the first element, one of the second element, none of the third element, and three of the fourth element of the set.

Theorem 2 There are $C(n+r-1,r) = C(n+r-1,n-1) = \frac{(n+r-1)!}{n!(r-1)!}$ r-combinations from a set with n elements when repetition of elements is allowed.

Proof:

As we have seen, each different list containing $n-1$ bars and r stars corresponds to an r-combination of the set with n elements, when repetition is allowed. The number of such lists is $C(n-1+r, r)$, because each list corresponds to a choice of the r positions to place the r stars from the $n-1+r$ positions that contain r stars and $n-1$ bars.

The number of such lists is also equal to $C(n-1+r, n-1)$, because each list corresponds to a choice of the $n-1$ positions to place the $n-1$ bars. ■

Permutations with indistinguishable objects

Example 3

How many distinct rearrangements are there of the letters in the word ABRACADABRA?

Solution:

Because some of the letters of ABRACADABRA are the same, the answer is not given by the number of permutations of eleven letters.

There are 11! permutations of the letters A, B, R, A, C, A, D, A, B, R and A. Since the five A's, two B's and two R's are indistinguishable, there are

$$\frac{11!}{5!\,2!\,2!} = 83160$$

different rearrangements. ■

Permutations with indistinguishable objects

Theorem 3

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\ldots n_k!}.$$

Rearrangement theorem

Theorem 4

Let n be a nonnegative integer. Then

$$(x_1 + x_2 + \cdots + x_k)^n$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1!\, n_2! \ldots n_k!} {x_1}^{n_1} {x_2}^{n_2} \ldots {x_k}^{n_k}$$

Proof:

The terms in the product $(x_1 + x_2 + \cdots + x_k)^n$ when it is expanded are of the form ${x_1}^{n_1} {x_2}^{n_2} \ldots {x_k}^{n_k}$, $n_1 + n_2 + \cdots + n_k = n$.

Proof of the rearrangement theorem

- The terms in the product $(x_1 + x_2 + \cdots + x_k)^n$ when it is expanded are of the form ${x_1}^{n_1}{x_2}^{n_2} \ldots {x_k}^{n_k}$, $n_1 + n_2 + \cdots + n_k = n$.

To count the number of terms of the form

${x_1}^{n_1}{x_2}^{n_2} \ldots {x_k}^{n_k}$,

it is necessary to choose n_1 elements x_1's from n_1 sums $(x_1 + x_2 + \cdots + x_k)$,

it is necessary to choose n_2 elements $x_2's$ from remaining $n - n_1$ sums $(x_1 + x_2 + \cdots + x_k)$,

… ,

it is necessary to choose n_k elements x_k's from remaining $n - n_1 - \cdots - n_{k-1}$ sums $(x_1 + x_2 + \cdots + x_k)$.

Proof of the rearrangement theorem

To count the number of terms of the form $x_1{}^{n_1}x_2{}^{n_2} \ldots x_k{}^{n_k}$, it is necessary to choose n_1 elements x_1's from n_1 sums $(x_1 + x_2 + \cdots + x_k)$,

it is necessary to choose n_2 elements $x_2's$ from remaining $n - n_1$ sums $(x_1 + x_2 + \cdots + x_k)$,

… ,

it is necessary to choose n_k elements x_k's from remaining $n - n_1 - \cdots - n_{k-1}$ sums $(x_1 + x_2 + \cdots + x_k)$.

We can do it by

$$C(n, n_1)C(n - n_1, n_2) \ldots C(n - n_1 - \cdots - n_{k-1}, n_k)$$

$= \frac{n!}{n_1!n_2!\ldots n_k!}$ ways. ■

Rearrangement theorem

Example 4

Determine the coefficient of ab^2c^3 in the expansion $(a+b+c)^6$.

Solution:

The coefficient in the expansion $(a+b+c)^6$ is

$$\frac{6!}{1!\,2!\,3!} = 60.$$