Introduction to Vectors

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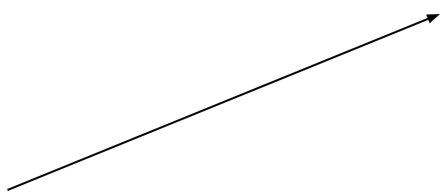
What are Vectors?

- Vectors are pairs of a direction and a magnitude. We usually represent a vector with an arrow:



- The direction of the arrow is the direction of the vector, the length is the magnitude.

Vectors in R^n

$n = 1$ R^1-space = set of all real numbers

(R^1-space can be represented geometrically by the x-axis)

$n = 2$ R^2-space = set of all ordered pair of real numbers (x_1, x_2)

(R^2-space can be represented geometrically by the xy-plane)

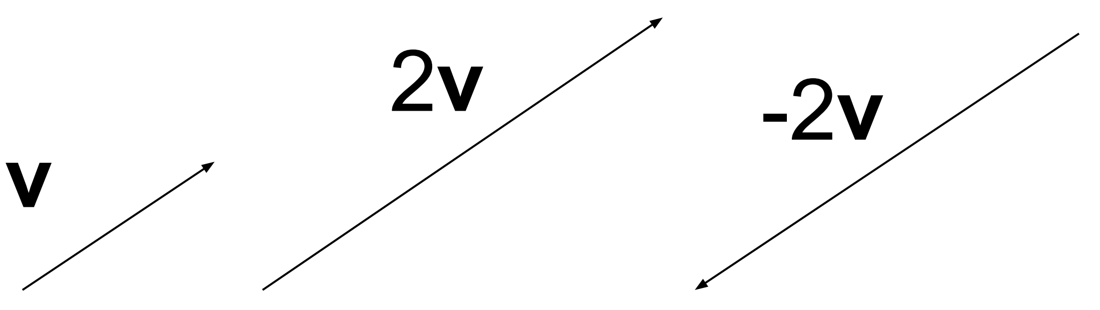
$n = 3$ R^3-space = set of all ordered triple of real numbers (x_1, x_2, x_3)

(R^3-space can be represented geometrically by the xyz-space)

$n = 4$ R^4-space = set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

Multiples of Vectors

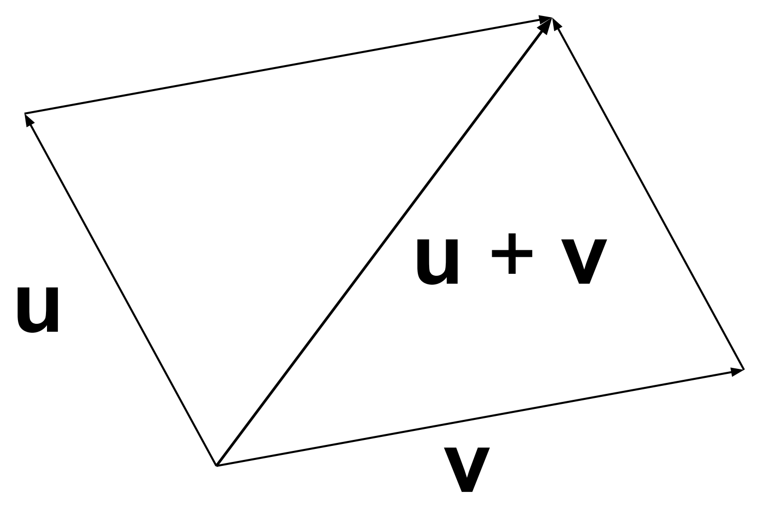
Given a real number *c*, we can multiply a vector by *c* by multiplying its magnitude by *c*:



Notice that multiplying a vector by a negative real number reverses the direction.

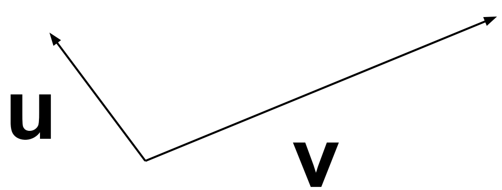
Adding Vectors

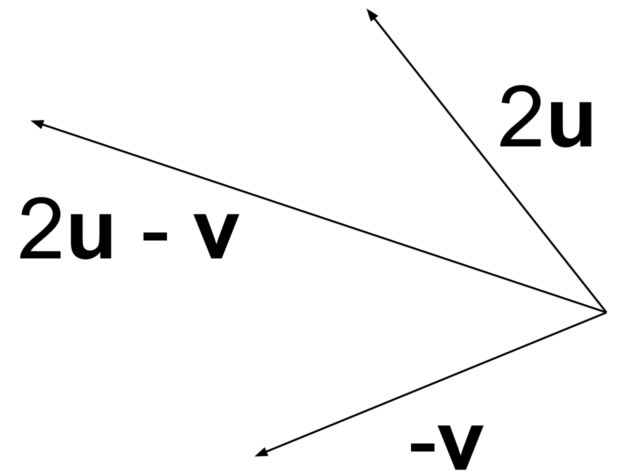
Two vectors can be added using the **Parallelogram Law**



Combinations

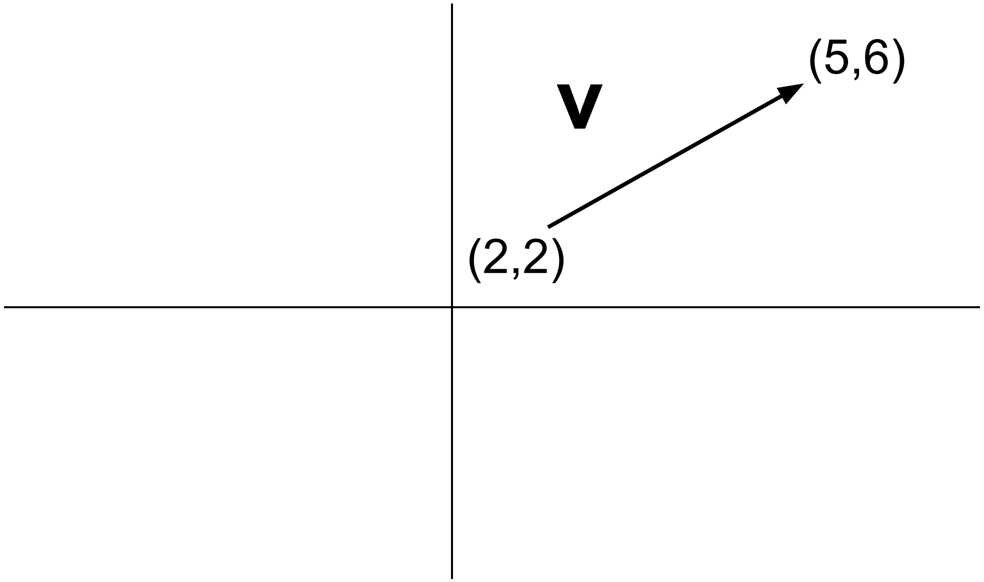
These operations can be combined.





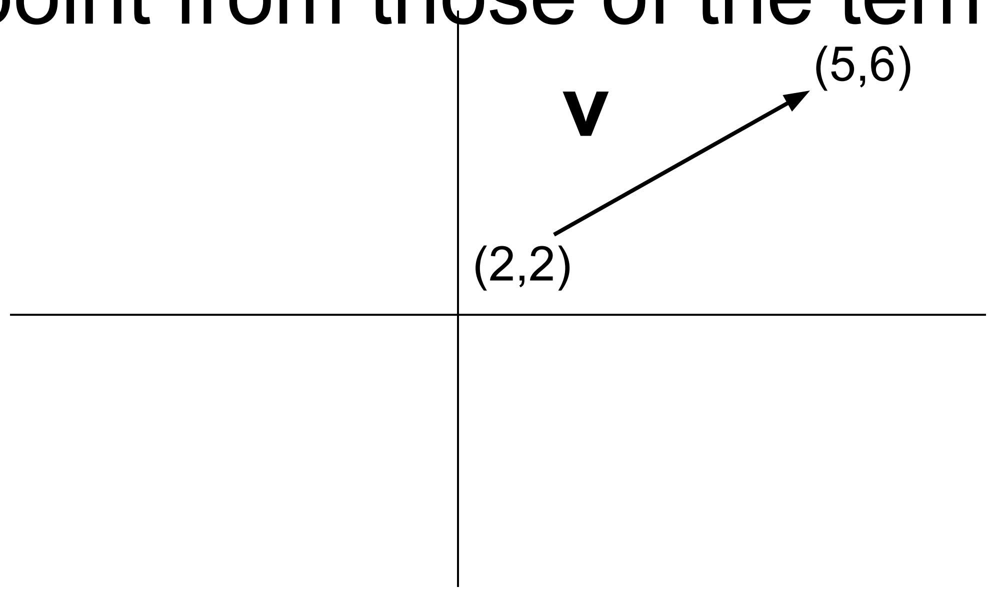
Components

To do computations with vectors, we place them in the plane and find their **components**.



Components

The initial point is the tail, the head is the terminal point. The components are obtained by subtracting coordinates of the initial point from those of the terminal point.

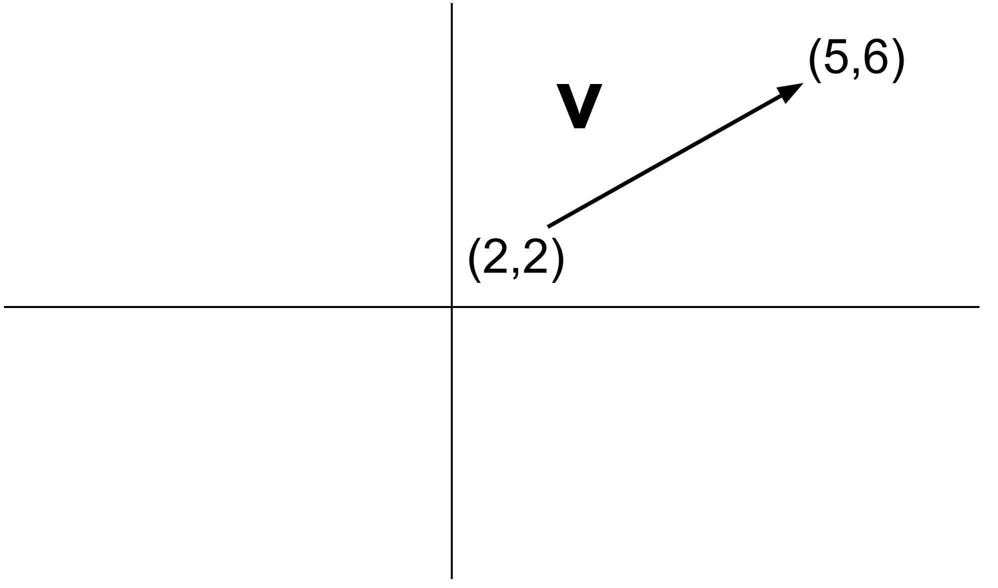


Components

The first component of **v** is 5 -2 = 3.

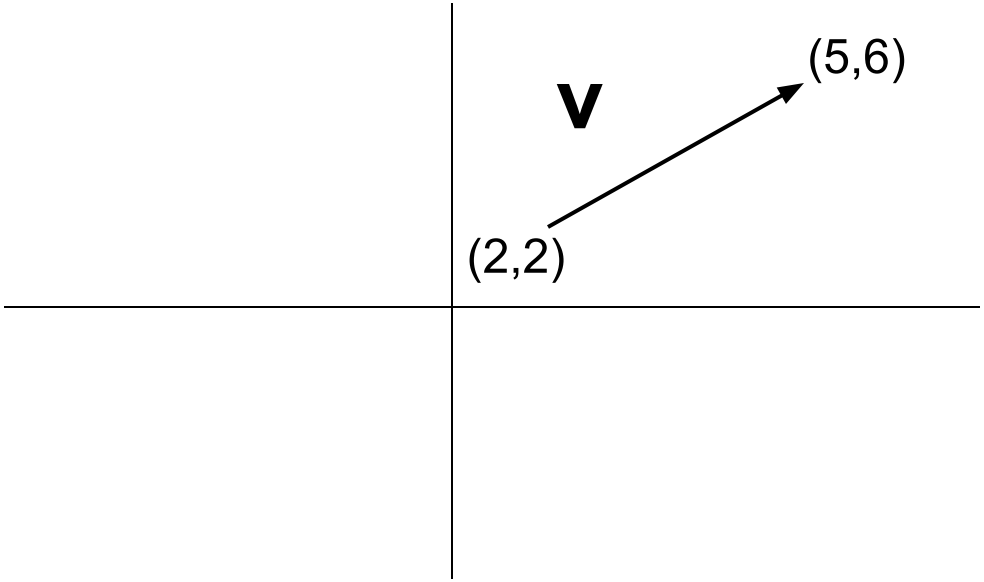
The second is 6 -2 = 4.

We write **v** = <3,4>



Magnitude

The magnitude of the vector is the length of the segment, it is written ||**v**||.



Scalar Multiplication

Once we have a vector in component form, the arithmetic operations are easy.

To multiply a vector by a real number, simply multiply each component by that number.

Example: If **v** = <3,4>, -2**v** = <-6,-8>

Addition

To add vectors, simply add their components.

For example, if **v** = <3,4> and **w** = <-2,5>, then **v** + **w** = <1,9>.

Other combinations are possible.
For example: 4**v** – 2**w** = <16,6>.

Unit Vectors

A **unit vector** is a vector with magnitude 1.

Given a vector **v**, we can form a unit vector by multiplying the vector by 1/||**v**||.

For example, find the unit vector in the direction <3,4>:

Special Unit Vectors

A vector such as <3,4> can be written as 3<1,0> + 4<0,1>.

For this reason, these vectors are given special names: **i** = <1,0> and **j** = <0,1>.

A vector in component form **v** = <a,b> can be written a**i** + b**j**.

Dot Product of Vectors

Let ***u*** and ***v*** be two nonzero vectors in 2-space or 3-space, and assume these vectors have been positioned so that their initial points coincide. By the ***angle between u and v***, we shall mean the angle θ determined by ***u*** and ***v*** that satisfies $0 \le \theta \le \pi$ (Figure 3.3.1).

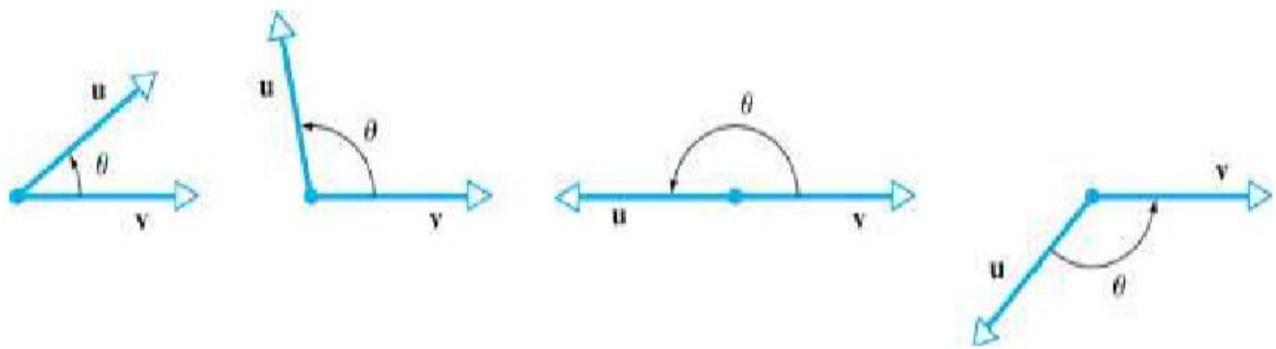


Figure 3.3.1

The angle θ between ***u*** and ***v*** satisfies $0 \le \theta \le \pi$.

DEFINITION

If ***u*** and ***v*** are vectors in 2-space or 3-space and θ is the angle between ***u*** and ***v***, then the ***dot product*** or ***Euclidean inner product*** $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta & \text{if } \mathbf{u} \ne \mathbf{0} \text{ and } \mathbf{v} \ne \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases} \tag{1}$$

Three dimensional space:

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1\mathbf{i}\cdot\mathbf{i} + a_1b_2\mathbf{i}\cdot\mathbf{j} + a_1b_3\mathbf{i}\cdot\mathbf{k} + a_2b_1\mathbf{j}\cdot\mathbf{i} + a_2b_2\mathbf{j}\cdot\mathbf{j} + a_2b_3\mathbf{j}\cdot\mathbf{k} + a_3b_1\mathbf{k}\cdot\mathbf{i} + a_3b_2\mathbf{k}\cdot\mathbf{j} + a_3b_3\mathbf{k}\cdot\mathbf{k}$$

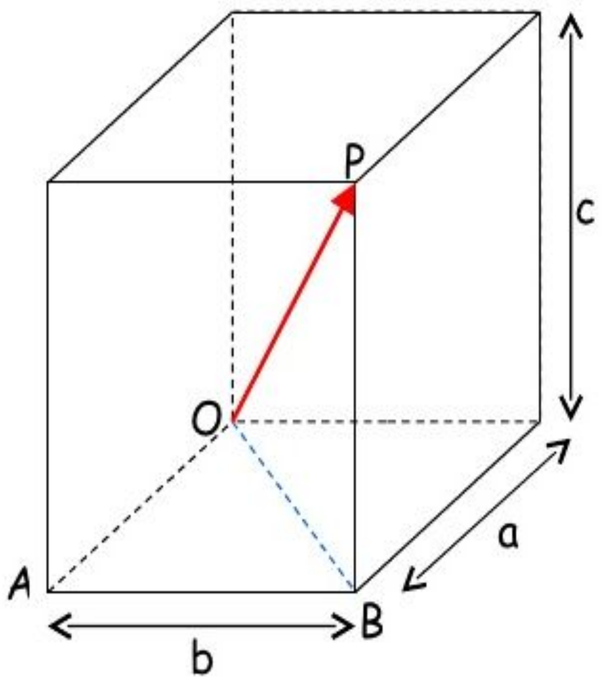
The unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have length 1 and are at 90° to each other and so any unit vector when scalar product combined with itself will give:

$$\mathbf{i}\cdot\mathbf{i} = 1 \times 1 \times \cos 0° = 1$$

Whilst any unit vector when scalar product combined with a different one will give: $\mathbf{i}\cdot\mathbf{j} = 1 \times 1 \times \cos 90° = 0$

Therefore $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Three dimensional space:



From the diagram the length of OP in terms of the side lengths can be determined as follows:

$OP^2 = OB^2 + BP^2$ and $OB^2 = OA^2 + AB^2$

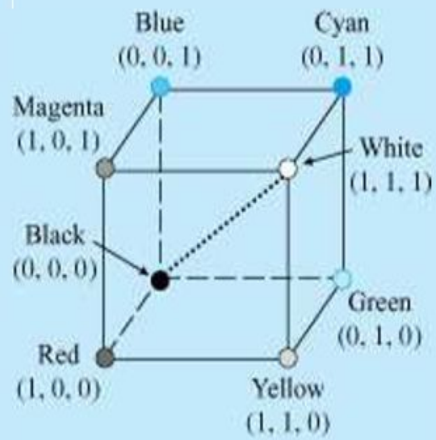
Thus $OP^2 = OA^2 + AB^2 + BP^2$

$OP^2 = a^2 + b^2 + c^2 \qquad OP = \sqrt{a^2 + b^2 + c^2}$

For our two vectors: $a = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $b = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Using, $\cos\theta = \dfrac{a \cdot b}{ab} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \times \sqrt{b_1^2 + b_2^2 + b_3^2}}$

Application to Computer Color Models



Colors on computer monitors are commonly based on what is called the ***RGB color model***. Colors in this system are created by adding together percentages of the primary colors red (R), green (G), and blue (B). One way to do this is to identify the primary colors with the vectors

$$\begin{aligned} \mathbf{r} &= (1, 0, 0) && \text{(pure red)},\\ \mathbf{g} &= (0, 1, 0) && \text{(pure green)},\\ \mathbf{b} &= (0, 0, 1) && \text{(pure blue)} \end{aligned}$$

in R^3 and to create all other colors by forming linear combinations of ***r***, ***g***, and ***b*** using coefficients between 0 and 1, inclusive; these coefficients represent the percentage of each pure color in the mix. The set of all such color vectors is called ***RGB space*** or the ***RGB color cube***. Thus, each color vector ***c*** in this cube is expressible as a linear combination of the form

$$\begin{aligned} \mathbf{c} &= c_1\mathbf{r} + c_2\mathbf{g} + c_3\mathbf{b}\\ &= c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1)\\ &= (c_1, c_2, c_3) \end{aligned}$$

Properties of Vector Arithmetic

If **u**, **v**, *and* **w** *are vectors in 2- or 3-space and k and l are scalars, then the following relationships hold.*

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(c) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(d) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

(e) $k(l\mathbf{u})=(kl)\mathbf{u}$

(f) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

(g) $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$

(h) $1\mathbf{u}=\mathbf{u}$

Proof of part (b) (analytic) We shall give the proof for vectors in 3-space; the proof for 2-space is similar. If $\mathbf{u}=(u_1,u_2,u_3)$, $\mathbf{v}=(v_1,v_2,v_3)$, and $\mathbf{w}=(w_1,w_2,w_3)$, then

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= [(u_1,u_2,u_3)+(v_1,v_2,v_3)]+(w_1,w_2,w_3)\\ &= (u_1+v_1,u_2+v_2,u_3,\ +v_3)+(w_1,w_2,w_3)\\ &= ([u_1+v_1]+w_1,[u_2+v_2]+w_2,[u_3+v_3]+w_3)\\ &= (u_1+[v_1+w_1],u_2+[v_2+w_2],u_3+[v_3+w_3])\\ &= (u_1,u_2,u_3)+(v_1+w_1,v_2+w_2,v_3+w_3)\\ &= \mathbf{u}+(\mathbf{v}+\mathbf{w})\end{aligned}$$

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Proof of part (b) (geometric) Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be represented by $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$ as shown in Figure 3.2.1. Then

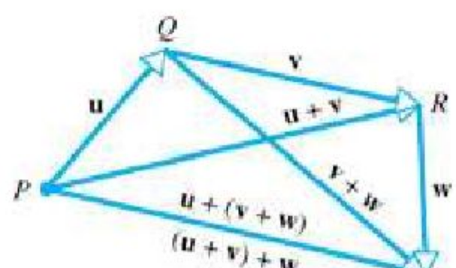
$$\mathbf{v}+\mathbf{w}=\overrightarrow{QS} \qquad \text{and} \qquad \mathbf{u}+(\mathbf{v}+\mathbf{w})=\overrightarrow{PS}$$

Also,

$$\mathbf{u}+\mathbf{v}=\overrightarrow{PR} \qquad \text{and} \qquad (\mathbf{u}+\mathbf{v})+\mathbf{w}=\overrightarrow{PS}$$

Therefore,

$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$$



An Orthogonal Projection

In many applications it is of interest to "decompose" a vector $\boldsymbol{u}$ into a sum of two terms, one parallel to a specified nonzero vector $\boldsymbol{a}$ and the other perpendicular to $\boldsymbol{a}$. If $\boldsymbol{u}$ and $\boldsymbol{a}$ are positioned so that their initial points coincide at a point Q, we can decompose the vector $\boldsymbol{u}$ as follows (Figure 3.3.6): Drop a perpendicular from the tip of $\boldsymbol{u}$ to the line through $\boldsymbol{a}$, and construct the vector $\mathbf{w}_1$ from Q to the foot of this perpendicular. Next form the difference

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$$

As indicated in Figure 3.3.6, the vector $\mathbf{w}_1$ is parallel to $\boldsymbol{a}$, the vector $\mathbf{w}_2$ is perpendicular to $\boldsymbol{a}$, and

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$$

The vector $\mathbf{w}_1$ is called the ***orthogonal projection of u on a*** or sometimes the ***vector component of u along a***. It is denoted by

$$\mathrm{proj}_{\mathbf{a}}\mathbf{u} \tag{7}$$

The vector $\mathbf{w}_2$ is called the ***vector component of u orthogonal to a***. Since we have $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$, this vector can be written in notation 7 as

$$\mathbf{w}_2 = \mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u}$$

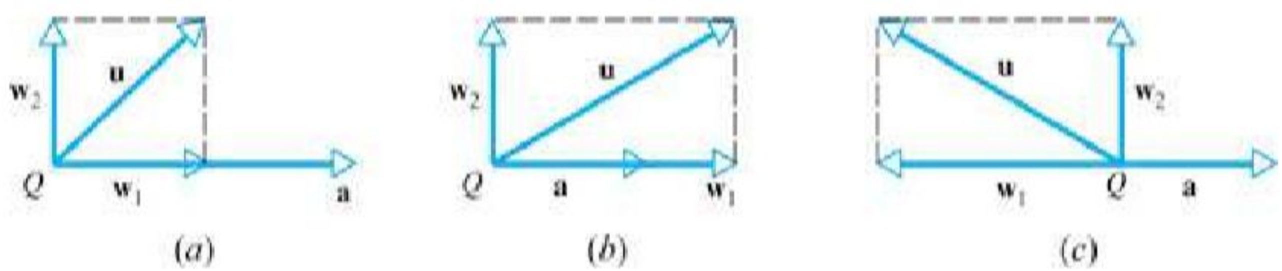


Figure 3.3.6

The vector $\boldsymbol{u}$ is the sum of $\mathbf{w}_1$ and $\mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\boldsymbol{a}$ and $\mathbf{w}_2$ is perpendicular to $\boldsymbol{a}$.

THEOREM 3.3.3

If $\mathbf{u}$ *and* $\mathbf{a}$ *are vectors in 2-space or 3-space and if* $\mathbf{a} \neq \mathbf{0}$*, then*

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \qquad \text{(vector component of } \mathbf{u} \text{ along } \mathbf{a}\text{)}$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} \qquad \text{(vector component } \mathbf{u} \text{ orthogonal to } \mathbf{a}\text{)}$$

EXAMPLE 6 Vector Component of *u* Along *a*

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.

Solution

$$\mathbf{u} \cdot \mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$$
$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus the vector component of ***u*** along ***a*** is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \tfrac{15}{21}(4, -1, 2) = \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right)$$

and the vector component of ***u*** orthogonal to ***a*** is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2, -1, 3) - \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right) = \left(-\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{11}{7}\right)$$

As a check, the reader may wish to verify that the vectors $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$ and ***a*** are perpendicular by showing that their dot product is zero.