Graphs

Irina Prosvirnina

- Eulerian paths
- Hamiltonian cycles

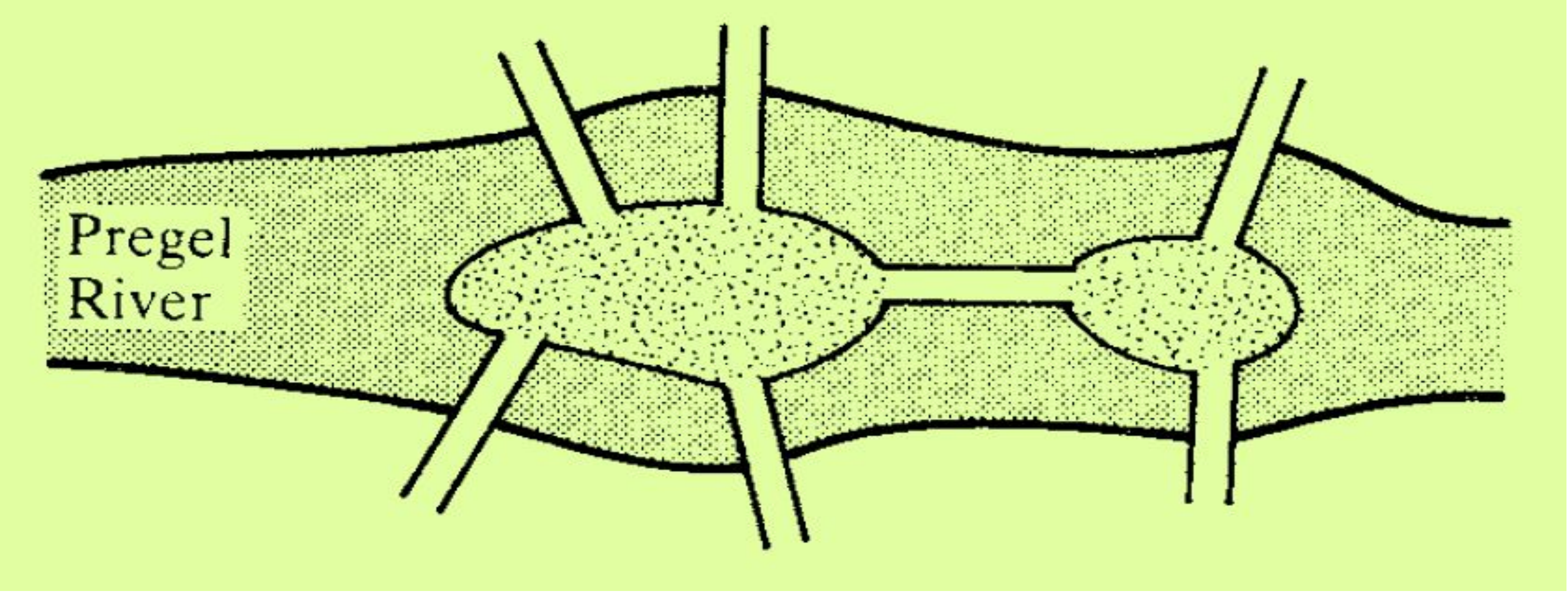
Eulerian paths

- We have mentioned Euler’s 1736 paper which marked the birth of graph theory.

This paper developed a theory which was able to solve the so-called Königsberg Bridge problem, which is the following.

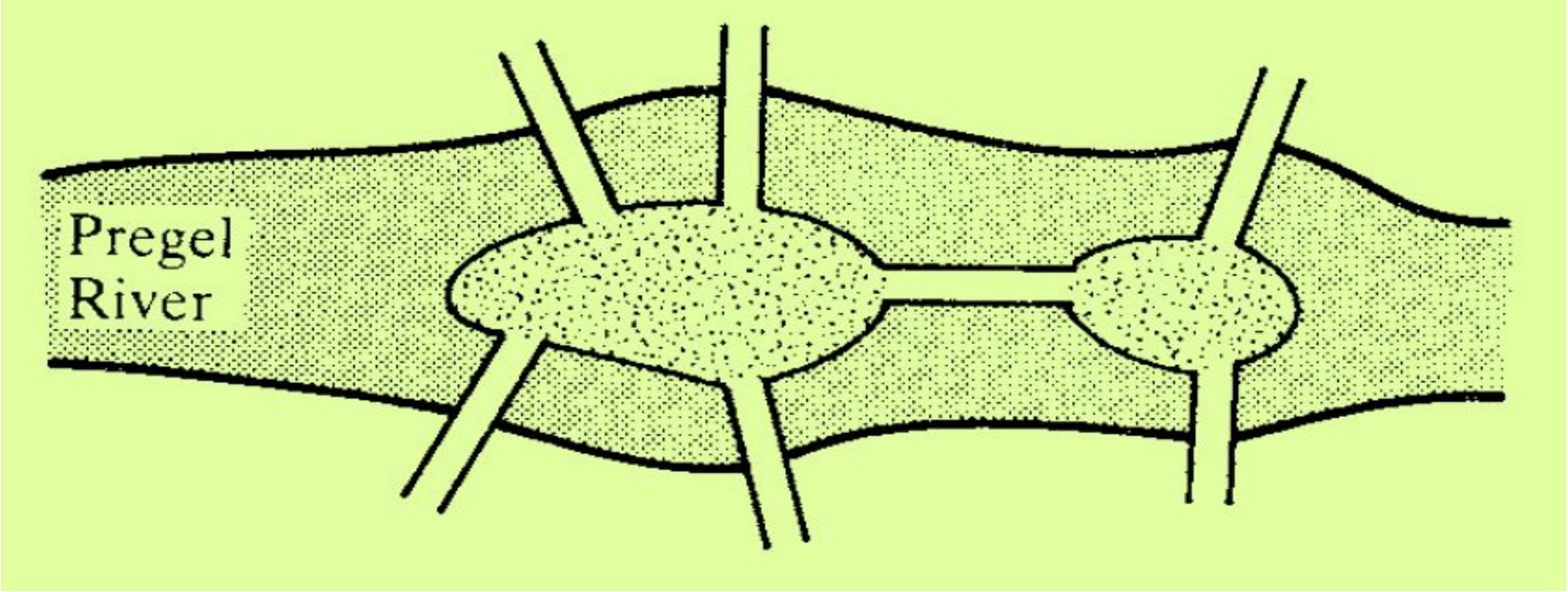
Eulerian paths

- The Pregel River flows through the town of Königsberg in Russia. There are two islands in the river, connected to the banks and each other by bridges as shown in the figure.



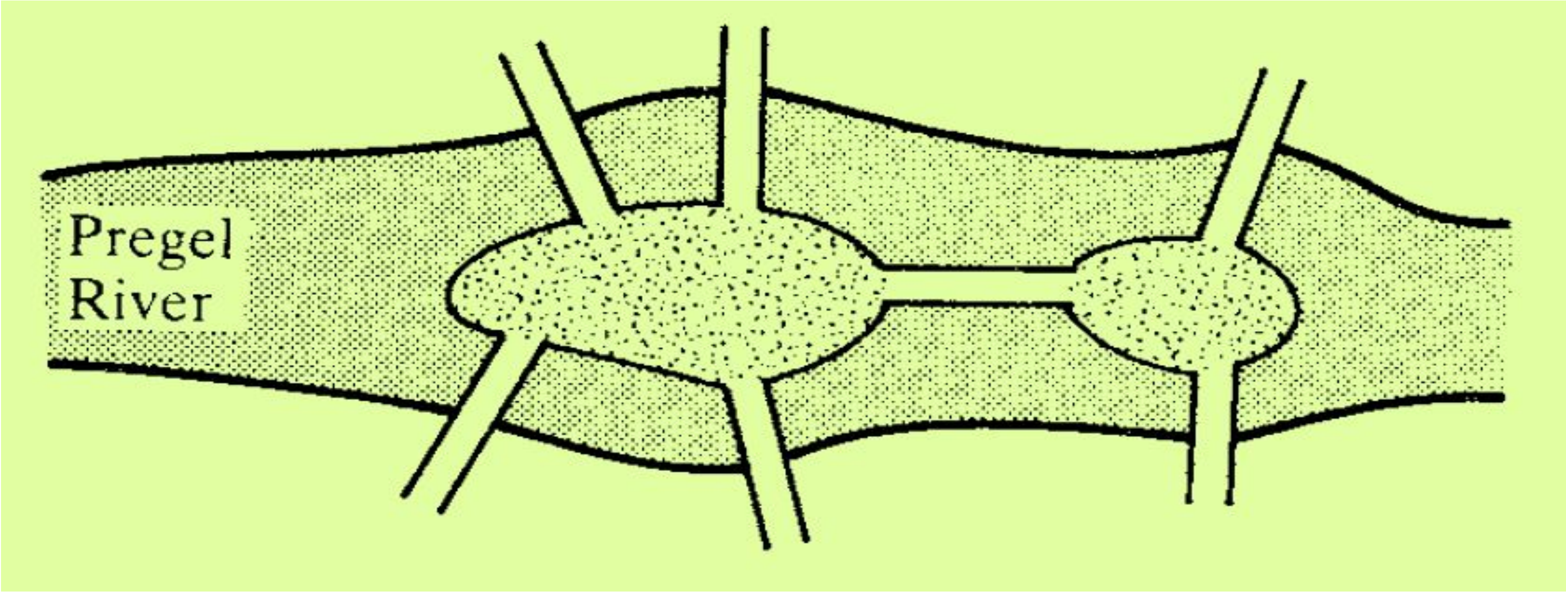
Eulerian paths

- The problem for the citizens of Königsberg was whether there was a walk, beginning on one of the banks or islands, which took in every bridge exactly once and finished back at the starting position.



Eulerian paths

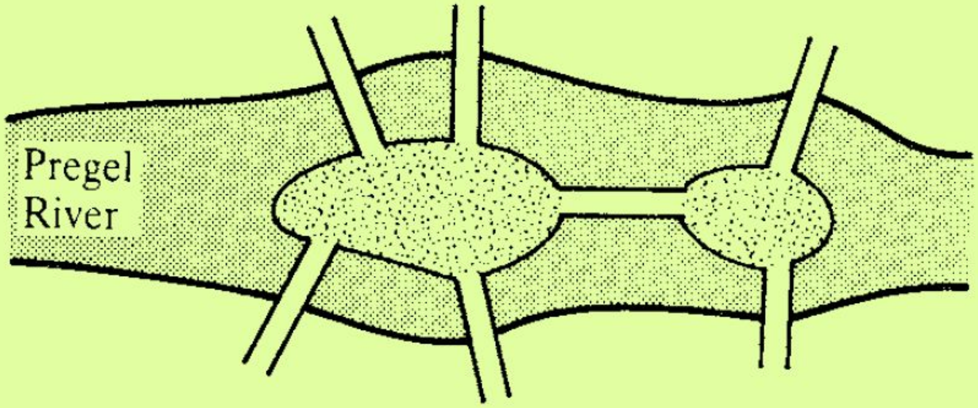
They were unable to find such a walk; the problem was either to find such a walk or to show that none existed.



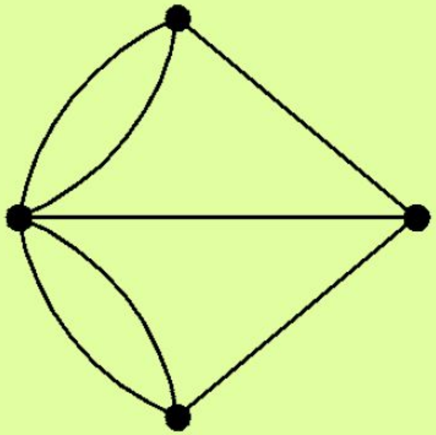
Eulerian paths

- Euler first represented the essential features of Königsberg 's geography by a graph, as illustrated in figure (b).

Each of the river banks and islands is represented by a vertex with the edges corresponding to the connecting bridges.



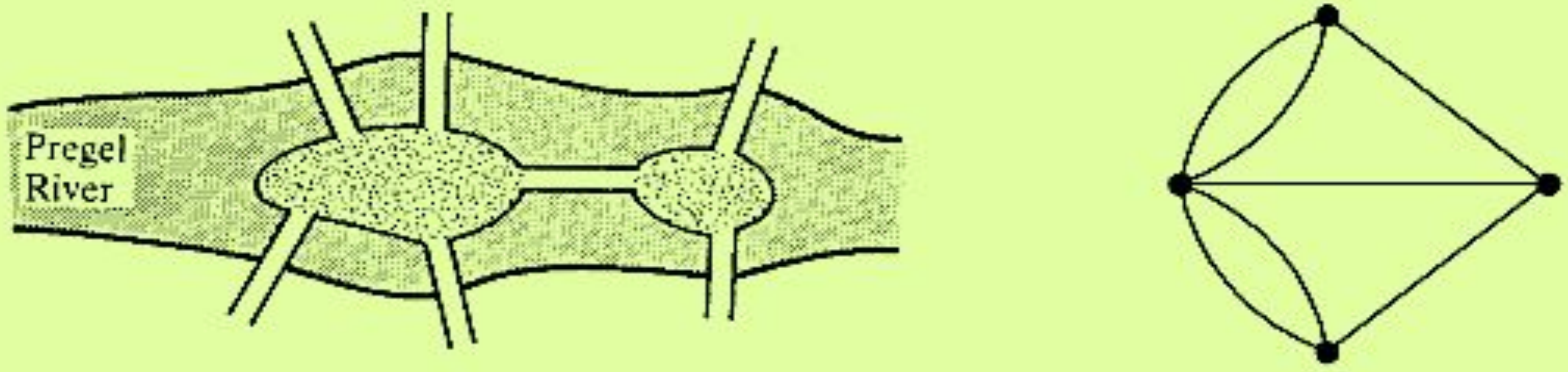
(a)



(b)

Eulerian paths

In graph-theoretic terms the question is whether there exists a closed path which includes all the edges of the graph.



(a) (b)

Eulerian paths

Definition 1

An **Eulerian path** in a graph Γ is a closed path which includes every edge of Γ.

A graph is said to be **Eulerian** if it has at least one Eulerian path.

Recall that in a path no edge can be traversed more than once.

Thus an Eulerian path includes every edge exactly once, although, of course, vertices may be visited more than once.

Eulerian paths

Theorem 1

A connected graph Γ is Eulerian if and only if every vertex has even degree.

? If a connected graph Γ is Eulerian then every vertex has even degree.

Proof

Suppose that Γ is connected and has an Eulerian path.

Since Γ is connected, the vertex sequence of the Eulerian path contains every vertex.

Whenever the path passes through a vertex it contributes two to its degree (one from the edge 'going in to' and one from the edge 'coming out from' the vertex).

Since every edge occurs exactly once in the path, every vertex must have even degree. ■

? If Γ is connected and every vertex has even degree then Γ is Eulerian.

Proof

One method of proof is by induction on $|E|$, the number of edges of Γ.

The inductive step is outlined below.

Firstly, choose any vertex of v of Γ and a closed path P beginning and ending at v.

If P contains every edge of Γ we are finished; otherwise remove all the edges of P to form a new graph Γ'.

? If Γ is connected and every vertex has even degree then Γ is Eulerian.

Proof

- This new graph Γ' may be disconnected.

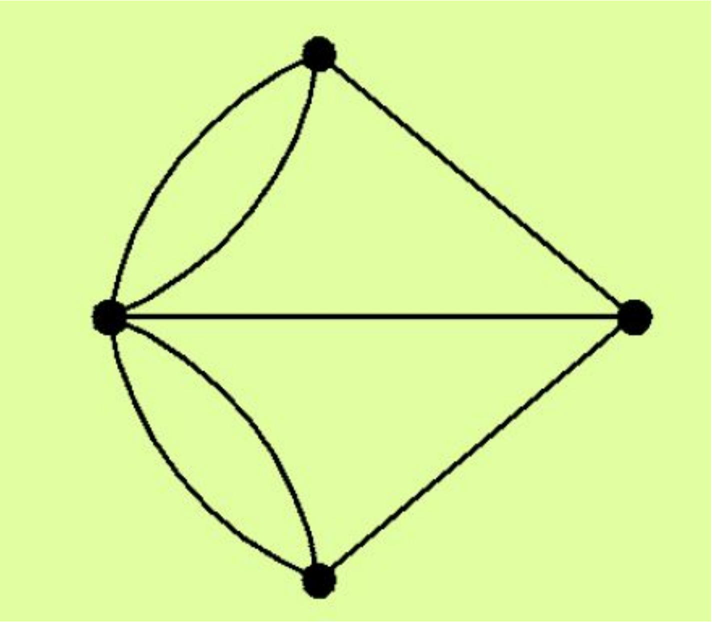
Consider each component of Γ' in turn and use the inductive hypothesis to obtain an Eulerian path in each of these components.

Finally use P and the Eulerian paths in each component of Γ' to piece together an Eulerian path for Γ. ■

Eulerian paths

- The people of Königsberg had not been able to find their Eulerian path for a very good reason – there isn’t one.

The graph representing the problem, is connected but fails the required condition. Every vertex, in fact, has odd degree.



Eulerian paths

Example 1

The complete graph K_n is $(n-1)$-regular – every vertex has degree $n-1$.

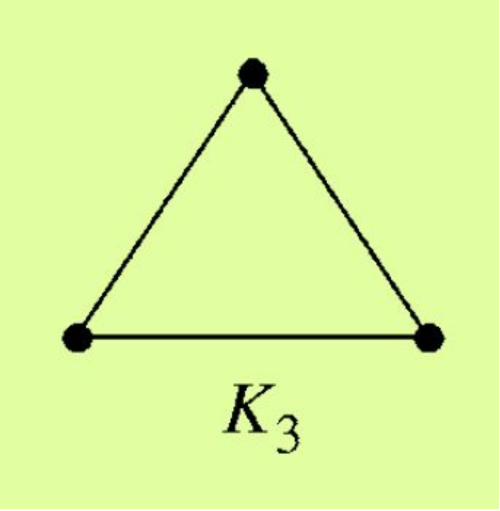
Since it is connected, K_n is Eulerian if and only if n is odd (so that $n-1$ is even).

Eulerian paths

Example 1

K_n is Eulerian if and only if n is odd (so that $n-1$ is even).

The graph K_3 has an obvious Eulerian path.

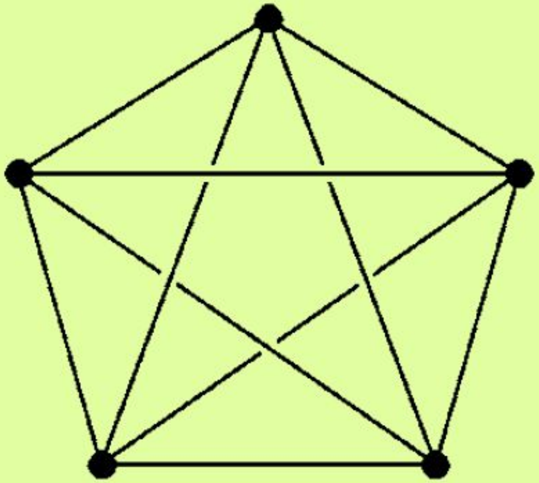


Eulerian paths

Example 1

K_n is Eulerian if and only if n is odd (so that $n-1$ is even).

Find an Eulerian path in K_5.
In fact, K_5 has 264 Eulerian paths.



K_5

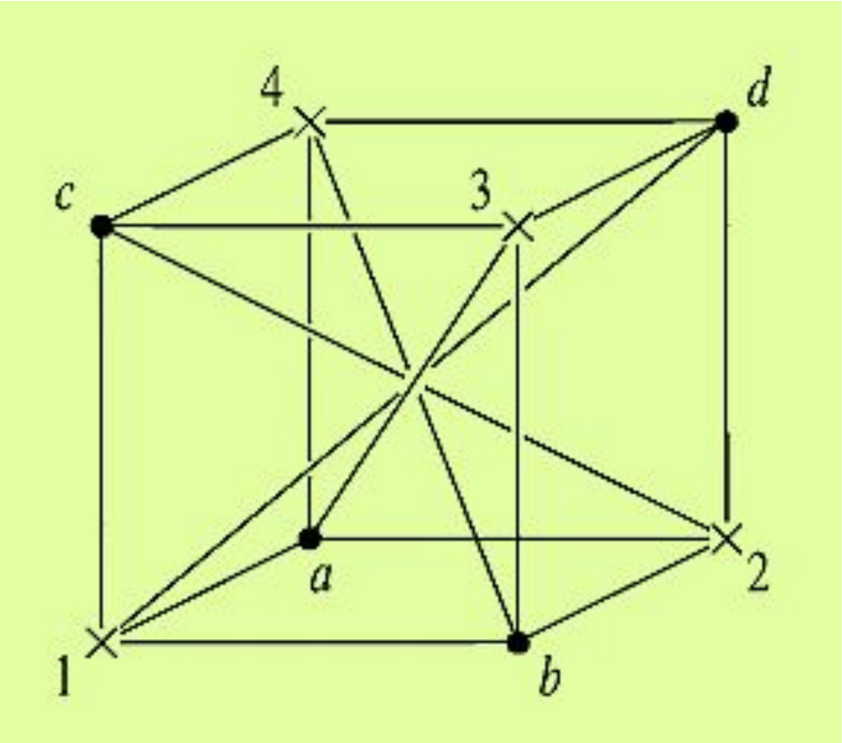
Eulerian paths

- The complete bipartite graph $K_{4,4}$ is represented in the figure.

The vertices have been partitioned into the sets {1, 2, 3, 4} and {a, b, c, d}.

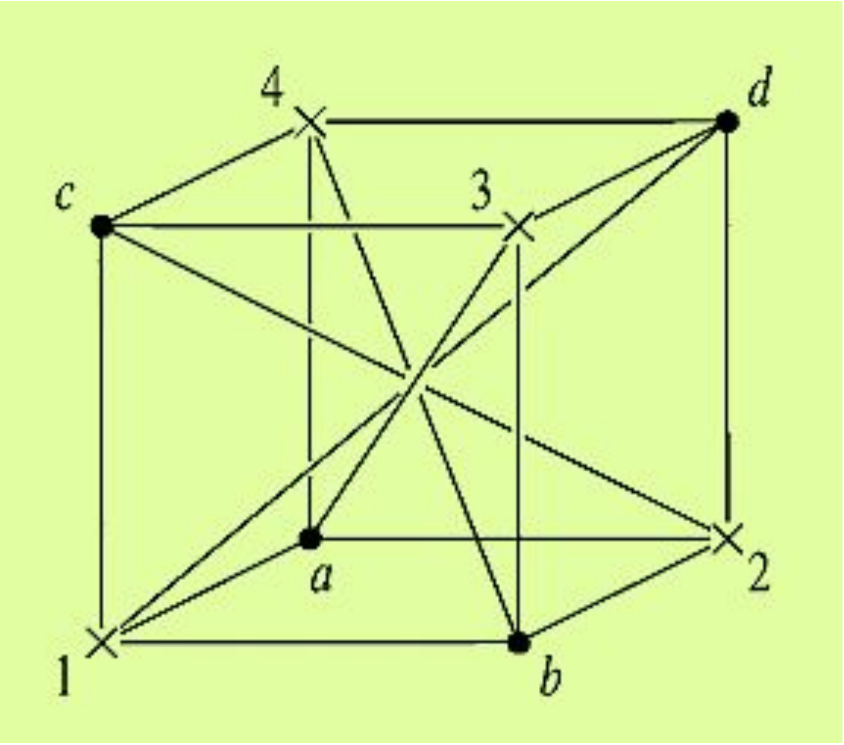
The graph is connected and every vertex has degree 4.

So $K_{4,4}$ is Eulerian by theorem 1.



Eulerian paths

- One Eulerian path beginning at the vertex 1 has the following vertex sequence: $1, a, 2, b, 3, c, 4, d, 1, c, 2, d, 3, a, 4, b, 1$.



Hamiltonian cycles

An Eulerian path seeks to travel along every edge of the graph (once) and return to the starting position.

An analogous problem is whether we can visit every vertex once, without travelling along any edge more than once, and return to the starting position.

This problem was considered by Hamilton (although he was not the first to do so) and his name is now associated with these paths.

Hamiltonian cycles

Definition 2

A **Hamiltonian cycle** in a graph is a cycle which passes once through every vertex.

A graph is **Hamiltonian** if it has a Hamiltonian cycle.

This terminology comes from a game, called the Icosian puzzle, invented in 1857 by the Irish mathematician Sir William Rowan Hamilton.

Hamiltonian cycles

Sir William Rowan Hamilton (1805 – 1865) was Ireland's most gifted mathematician-scientist. As a 22 year old undergraduate he was elected Professor of Astronomy and Astronomer Royal of Ireland.

In fact he made little contribution to astronomy; his most significant work was in mathematics and physics.

Sir William Rowan Hamilton (1805 – 1865)

Hamiltonian cycles

In 1843 he discovered the quaternions – a sort of generalized complex numbers – and he devoted most of the rest of his life to their study.

His name is also associated with the Hamiltonian energy operator used in physics, particularly wave mechanics.

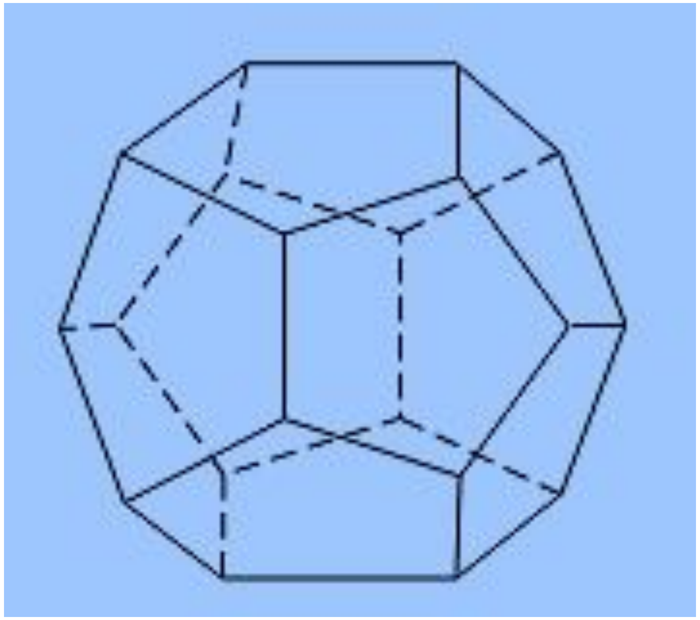
Sir William Rowan Hamilton

(1805 – 1865)

Hamiltonian cycles

The Icosian puzzle consisted of a wooden dodecahedron (a polyhedron with 12 regular pentagons as faces, as shown in the figure), with a peg at each vertex of the dodecahedron, and string.

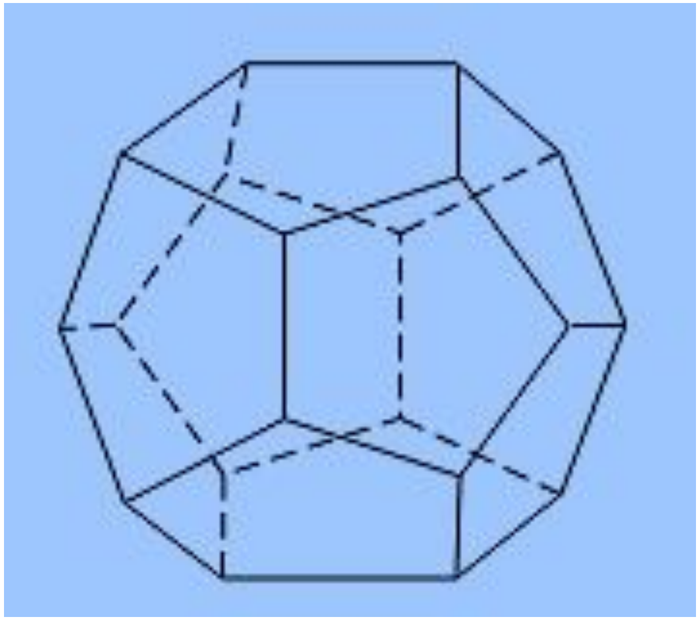
The 20 vertices of the dodecahedron were labeled with different cities in the world.



Hamiltonian cycles

The object of the puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end back at the first city.

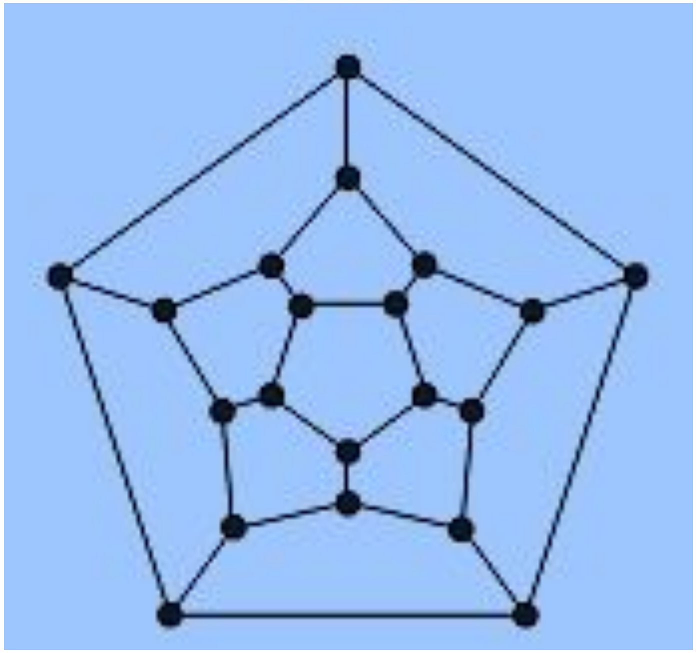
The circuit traveled was marked off using the string and pegs.



Hamiltonian cycles

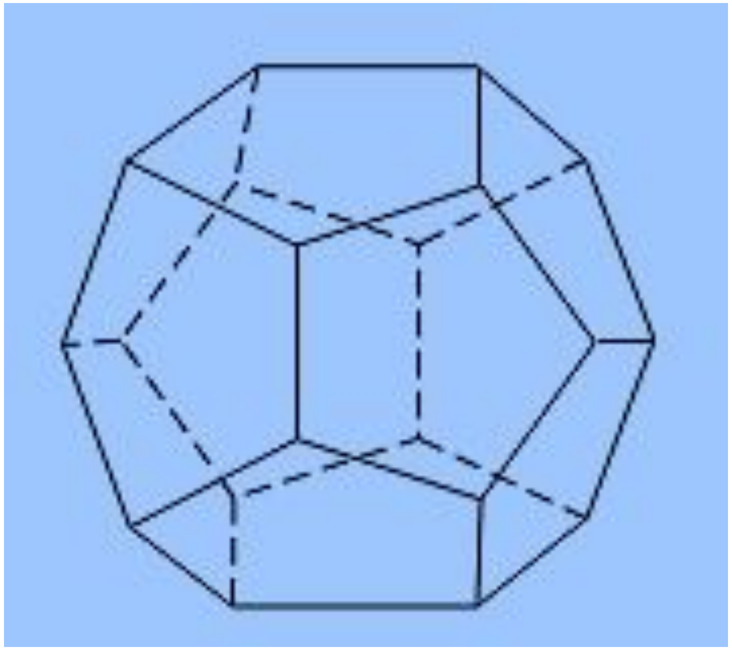
Because I cannot supply you with a wooden solid with pegs and string, we will consider the equivalent question:

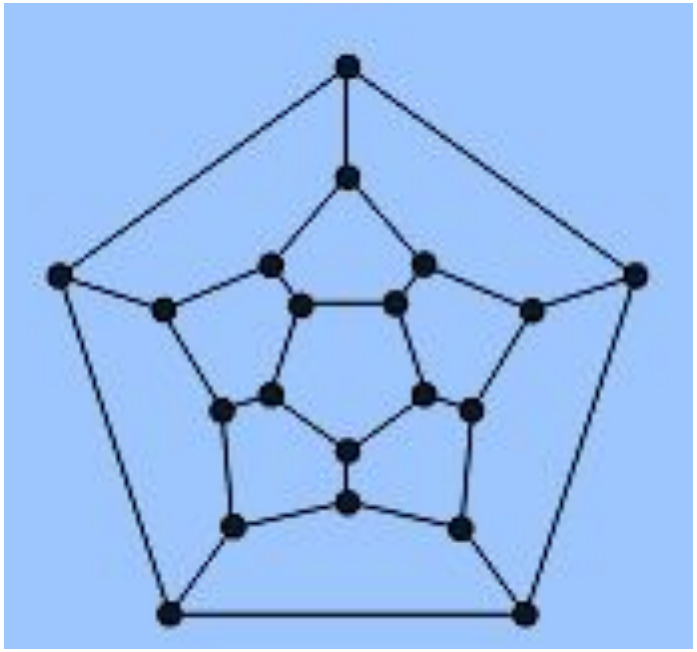
Is there a circuit in the graph shown in the figure that passes through each vertex exactly once?



Hamiltonian cycles

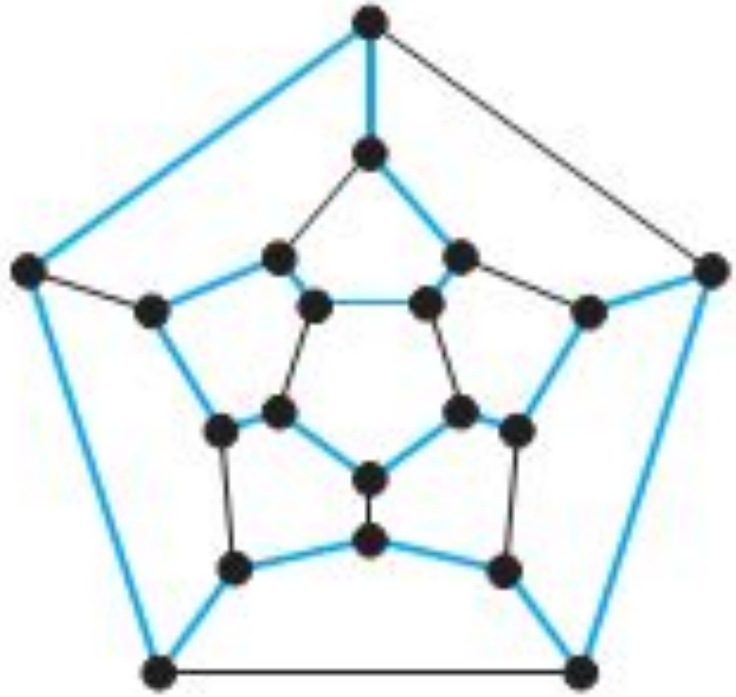
This solves the puzzle because this graph is isomorphic to the graph consisting of the vertices and edges of the dodecahedron.





Hamiltonian cycles

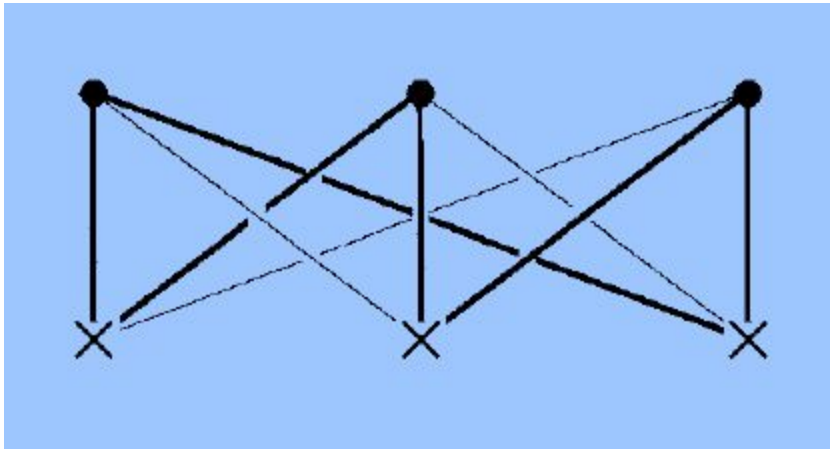
A solution of Hamilton's puzzle is shown in the figure.



Hamiltonian cycles

Example 1

The figure illustrates Hamiltonian cycle in the the complete bipartite graph $K_{3,3}$.



Hamiltonian cycles

Although Eulerian graphs have a simple characterization, the same is not true of Hamiltonian graphs.

Indeed after more than a century of study, no characterization of Hamiltonian graphs is known. (By a 'characterization' of Hamiltonian graphs we mean necessary and sufficient conditions for a graph to be Hamiltonian.)

Hamiltonian cycles

This remains one of the major unsolved problems of graph theory.

An obvious necessary condition is that the graph be connected.

Various sufficient conditions are also known; most require the graph to have ‘enough’ edges in some sense.

Hamiltonian cycles

- One of the simplest such results is the following

Theorem 1

If Γ is a connected simple graph with n (≥ 3) vertices and if the degree

$$deg(v) \geq \frac{1}{2}n$$

for every vertex v, then Γ is Hamiltonian.

The condition on the degrees, $deg(v) \geq \frac{1}{2}n$, is not a necessary condition for Γ to be Hamiltonian, so a graph can be Hamiltonian without satisfying this condition.

Hamiltonian cycles

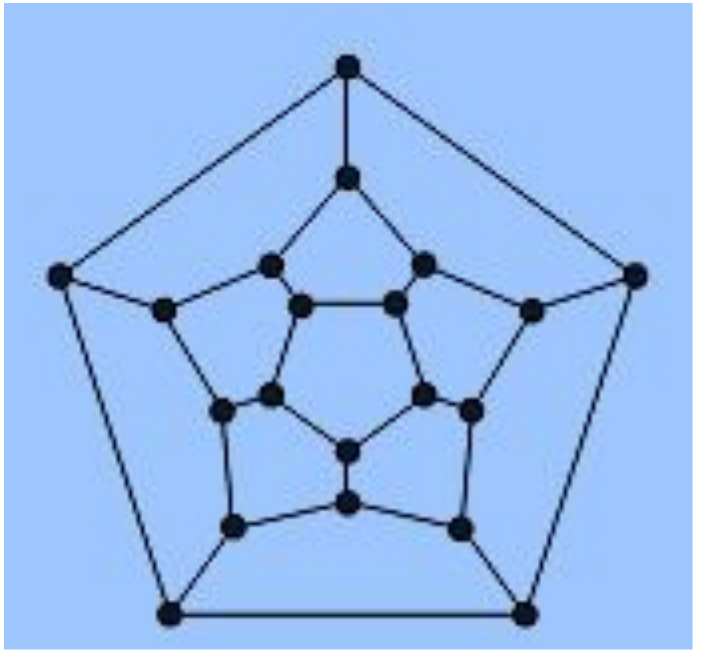
- We can see this by considering the dodecahedral graph.

The graph has 20 vertices, every vertex has degree 3 $(3 < 10)$, but it is still Hamiltonian.

Theorem 1
If Γ is a connected simple graph with n (≥ 3) vertices and if the degree

$$deg(v) \geq \frac{1}{2}n$$

for every vertex v, then Γ is Hamiltonian.



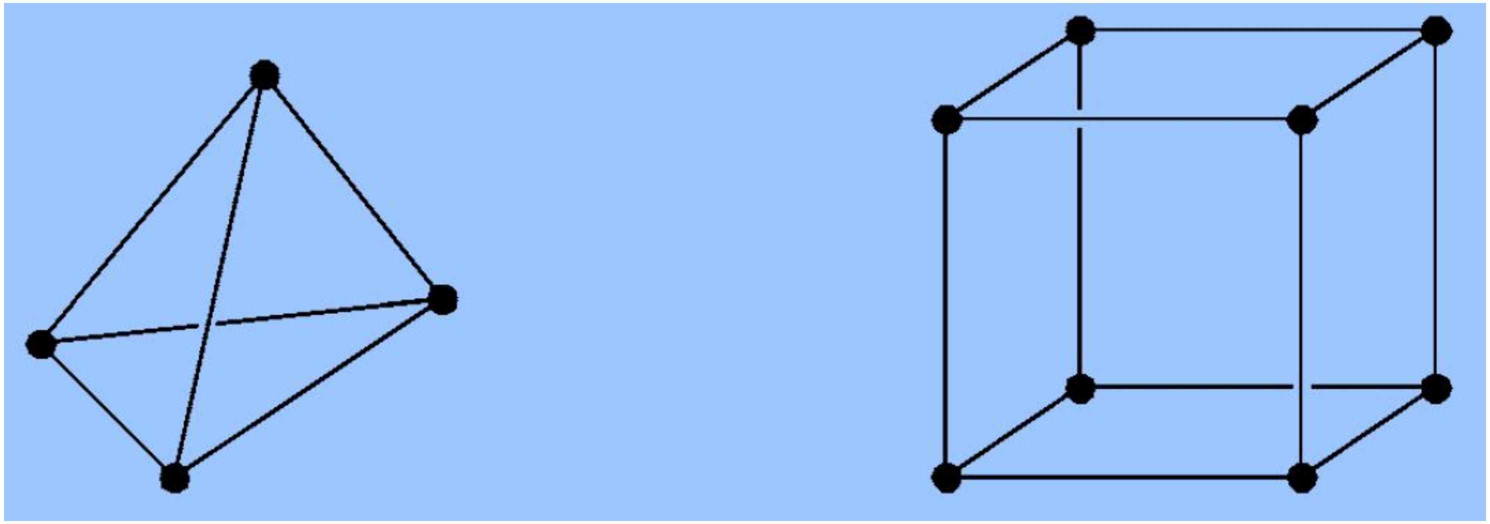
Hamiltonian cycles

In fact the graphs of each of the five regular solids has a Hamiltonian cycle.

Hamiltonian cycles

Exercise 1

Show that each of the graphs of the regular solids shown in the figure (tetrahedron K_4 and cube) is Hamiltonian.



Hamiltonian cycles

Exercise 1

Show that each of the graphs of the regular solids shown in the figure (octahedron and icosahedron) is Hamiltonian.