Lecture 8. Vectors

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- Linear dependence of vectors
- Basis on the plane and in space
- Decomposition of a vector by basis
- Direction cosines of a vector.
- Division of segment.

Linear combination

- Linear combination :

A vector $\mathbf{u}$ in a vector space V is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V if $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are real-number scalars

- Ex : Finding a linear combination

$$\mathbf{v}_1=(1,2,3) \quad \mathbf{v}_2=(0,1,2) \quad \mathbf{v}_3=(-1,0,1)$$

Prove (a) $\mathbf{w}=(1,1,1)$ is a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

(b) $\mathbf{w}=(1,-2,2)$ is not a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

Sol:

(a) $\mathbf{w}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1)&=c_1(1,2,3)+c_2(0,1,2)+c_3(-1,0,1)\\ &=(c_1-c_3,\ 2c_1+c_2,\ 3c_1+2c_2+c_3)\end{aligned}$$

$$\Rightarrow \begin{array}{rcl} c_1 \qquad\quad -c_3 &=& 1\\ 2c_1+\ c_2 \qquad\quad &=& 1\\ 3c_1+2c_2+c_3 &=& 1\end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

$$\overset{t=2}{\Rightarrow} \mathbf{w} = 3\mathbf{v}_1 - 5\mathbf{v}_2 + 2\mathbf{v}_3$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ This system has no solution since the third row means

$$0 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 7$$

$\Rightarrow$ $\mathbf{w}$ can not be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

Example 1. Decompose the vector $\overline{b} = \{8; 1\}$ by basis vectors $\overline{p} = \{1; 2\}$ and $\overline{q} = \{3; 1\}$.

Solution: Form the vector equation:

$x\overline{p} + y\overline{q} = \overline{b}$,

which can be written as a system of linear equations

$$\begin{cases} 1x + 3y = 8 \\ 2x + 1y = 1 \end{cases}$$

from the first equation express x

$$\begin{cases} x = 8 - 3y \\ 2x + y = 1 \end{cases}$$

Substitute x in the second equation

$$\begin{cases} x = 8 - 3y \\ 2(8 - 3y) + y = 1 \end{cases}$$

$$\begin{cases} x = 8 - 3y \\ 16 - 6y + y = 1 \end{cases}$$

$$\begin{cases} x = 8 - 3y \\ 5y = 15 \end{cases}$$

$$\begin{cases} x = 8 - 3y \\ y = 3 \end{cases}$$

$$\begin{cases} x = 8 - 3\cdot 3 \\ y = 3 \end{cases}$$

$$\begin{cases} x = -1 \\ y = 3 \end{cases}$$

Answer: $\overline{b} = -\overline{p} + 3\overline{q}$.

▪ Definitions of Linear Independence (L.I.) and Linear Dependence (L.D.) :

$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$: a set of vectors in a vector space V

For $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$

(1) If the equation has only the trivial solution $(c_1 = c_2 = \cdots = c_k = 0)$ then S (or $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$) is called **linearly independent**

(2) If the equation has a nontrivial solution (i.e., not all zeros), then S (or $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$) is called **linearly dependent** (The name of linear dependence is from the fact that in this case, there exist a $\mathbf{v}_i$ which can be represented by the linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots \mathbf{v}_k\}$ in which the coefficients are not all zero.

- Ex : Testing for linear independence

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \ \ c_2 + \qquad\ &= 0 \\ 3c_1 + 2c_2 + \ \ c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$(\text{or } \det(A) = -1 \neq 0, \text{ so there is only the trivial solution})$$

$$\Rightarrow S \text{ is (or } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \text{ are) linearly independent}$$

- EX: Testing for linear independence

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\{1+x-2x^2, 2+5x-x^2, x+x^2\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

i.e., $c_1(1+x-2x^2)+c_2(2+5x-x^2)+c_3(x+x^2)=0+0x+0x^2$

$$\begin{aligned} c_1+2c_2 \quad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G. E.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system has infinitely many solutions

(i.e., this system has nontrivial solutions, e.g., $c_1=2$, $c_2=-1$, $c_3=3$)

S is (or $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are) linearly dependent

Basis

- Basis :

V: a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

S is linearly independent

(For $\sum c_i \mathbf{v}_i = A\mathbf{x} = \mathbf{0}$, there is only the trivial solution $(\det(A) \neq 0)$,

- S is called a basis for V

Ex1: the **standard basis** vectors in R^3:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Ex 2: The nonstandard basis for R^2

Show that $S=\{\mathbf{v}_1,\mathbf{v}_2\}=\{(1,1),\ (1,-1)\}$ is a basis for R^2

(2) For $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{0}\ \Rightarrow\ \begin{cases} c_1+c_2=0 \\ c_1-c_2=0 \end{cases}$

Because the coefficient matrix of this system has a **nonzero determinant**, you know that the system has only the trivial solution. Thus you can conclude that S is linearly independent

According to the above two arguments, we can conclude that S is a (nonstandard) basis for R^2

Definition. The **direction cosines of the vector** $\overline{a}$ are the cosines of angles that the vector forms with the coordinate axes.

The direction cosines uniquely set the direction of vector.

Basic relation. To find the **direction cosines of the vector** $\overline{a}$ is need to divided the corresponding coordinate of vector by the length of the vector.

The coordinates of the unit vector is equal to its direction cosines.

Property of direction cosines. The sum of the squares of the direction cosines is equal to one.

Direction cosines of a vector formulas

Direction cosines of a vector formula for two-dimensional vector

In the case of the plane problem (Fig. 1) the direction cosines of a vector $\overline{a} = \{a_x ; a_y\}$ can be found using the following formula

$$\cos\alpha = \frac{a_x}{|\overline{a}|}; \quad \cos\beta = \frac{a_y}{|\overline{a}|}$$

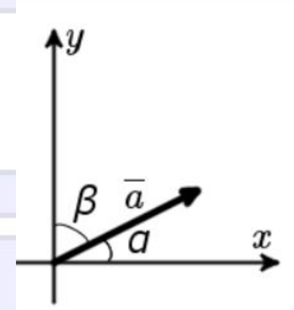


Fig. 1

Property:

$$\cos^2\alpha + \cos^2\beta = 1$$

Direction cosines of a vector formula for three-dimensional vector

In the case of the spatial problem (Fig. 2) the direction cosines of a vector $\overline{a} = \{a_x ; a_y ; a_z\}$ can be found using the following formula

$$\cos\alpha = \frac{a_x}{|\overline{a}|}; \quad \cos\beta = \frac{a_y}{|\overline{a}|}; \quad \cos\gamma = \frac{a_z}{|\overline{a}|}$$

Property:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

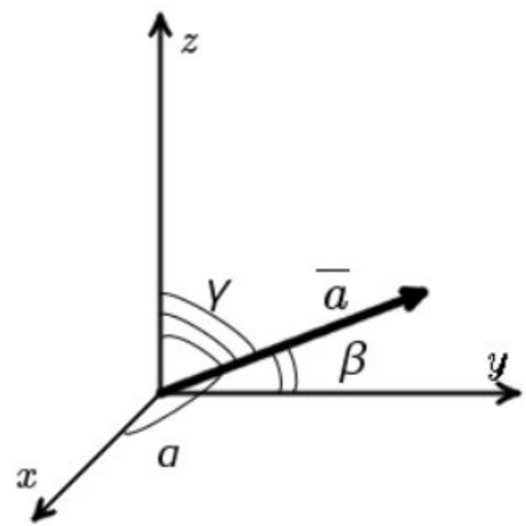


Fig. 2

Examples of plane tasks

Example 1. Find the direction cosines of the vector $\overline{a} = \{3; 4\}$.

Solution:

Calculate the length of vector $\overline{a}$:
$|\overline{a}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$

Calculate the direction cosines of the vector $\overline{a}$:

$$cos\ \alpha = \frac{a_x}{|\overline{a}|} = \frac{3}{5} = 0.6$$

$$cos\ \beta = \frac{a_y}{|\overline{a}|} = \frac{4}{5} = 0.8$$

Answer: direction cosines of the vector $\overline{a}$ is $cos\ \alpha = 0.6$, $cos\ \beta = 0.8$.

Example 2. Find the vector $\overline{a}$ if it length equal to 26, and direction cosines is $cos\ \alpha = 5/13$, $cos\ \beta = -12/13$.

Solution:

$a_x = |\overline{a}| \cdot cos\ \alpha = 26 \cdot 5/13 = 10$
$a_y = |\overline{a}| \cdot cos\ \beta = 26 \cdot (-12/13) = -24$

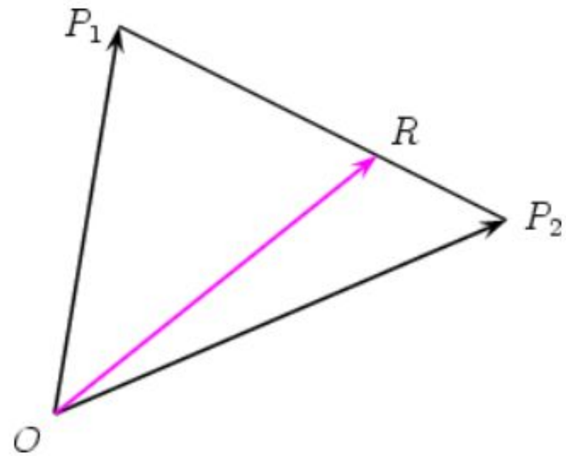
Answer: $\overline{a} = \{10; -24\}$.

5. Division of a line segment

Internal division of a line segment

With an assigned point O as origin, the position of any point P is given uniquely by the vector $\overrightarrow{OP}$, which is called the ***position vector*** of P relative to O.

Let P_1 and P_2 be any points, and let R be a point on the line P_1P_2 such that R divides the line segment P_1P_2 in the ratio $m : n$. That is, R is the point such that $\overrightarrow{P_1R} = \frac{m}{n}\overrightarrow{RP_2}$. Our task is to find the position vector of R (relative to O) in terms of the position vectors of P_1 and P_2.



As $\overrightarrow{P_1R} = \frac{m}{n}\overrightarrow{RP_2}$, we have $n\overrightarrow{P_1R} = m\overrightarrow{RP_2}$ and therefore

$$n(\overrightarrow{OR} - \overrightarrow{OP_1}) = m(\overrightarrow{OP_2} - \overrightarrow{OR}), \tag{1}$$

which rearranges to give

$$\boxed{\overrightarrow{OR} = \frac{n\overrightarrow{OP_1} + m\overrightarrow{OP_2}}{m+n}, \qquad m+n \neq 0.}$$

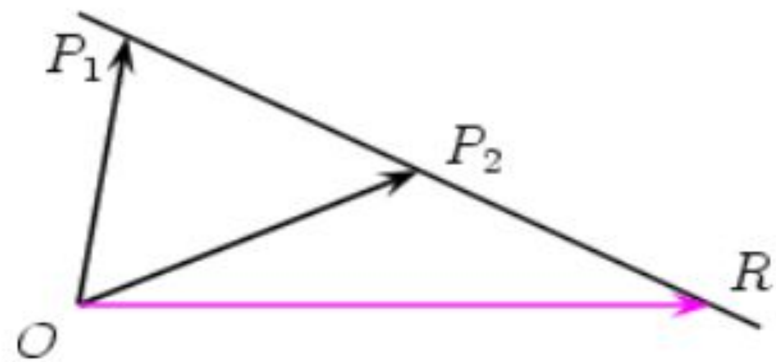
When m and n are both positive, the vectors $\overrightarrow{P_1R}$ and $\overrightarrow{RP_2}$ have the same direction, since $\overrightarrow{P_1R} = \frac{m}{n}\overrightarrow{RP_2}$. This corresponds to the situation where R lies between P_1 and P_2, as shown in the diagram above. R is then said to divide the line segment P_1P_2 ***internally*** in the ratio $m : n$.

Example 1

Given two points P_1 and P_2 in space find the point R dividing the line segment P_1P_2 in the ratio -2 : 1.

Solution

If R divides P_1P_2 in the ratio -2 : 1 then $\overrightarrow{P_1R}$ = -2$\overrightarrow{P_2R}$.



The position vector $\overrightarrow{OR}$ is then equal to

$$\overrightarrow{OR} = \frac{1\overrightarrow{OP_1} - 2\overrightarrow{OP_2}}{-2+1} = -\overrightarrow{OP_1} + 2\overrightarrow{OP_2}.$$

Applications of vectors

- https://www.machinelearningplus.com/nlp/cosine-similarity/
- http://www.cs.utoronto.ca/~strider/d18/LinAlg.pdf