Week 4: Geometric Modeling – Parametric Representation of Analytic Curves

Spring 2018, AUA

Intro to Geometric Modeling (GM)

The goal of CAD - efficient representation of the unambiguous and complete info about a design for the subsequent applications:

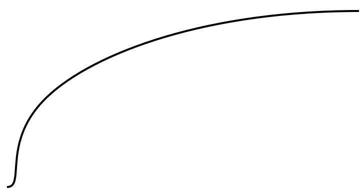
- mass property calculations
- mechanism analysis
- finite element analysis
- NC programming

Geometric modeling - defining geometric objects using computer compatible mathematical representation.

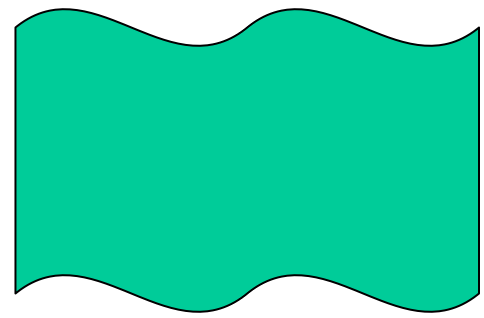
Mathematical representation learned in schools will not work.

As well as objects created in Word or Power Point or Photoshop.

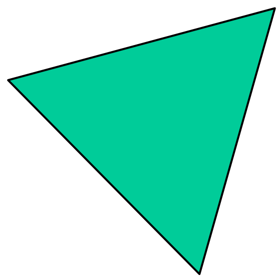
Objects of Representation



Curves



Surfaces



Solids

Standard form vs free-form

Domain of study – Computer Graphics

Types of Representation

Explicit Representation	Implicit Representation	Parametric Representation
$y = \pm\sqrt{r^2 - x^2}$	$x^2 + y^2 - r^2 = 0$	$x = r \cdot \cos(t)$ $y = r \cdot \sin(t)$
z=ax+by+cz+d	ex+fy+gz+h=0	x=a+bu+cw y=d+eu+fw z=g+hu+iw

The question is which one is computer compatible?

Advantages of PR

- **<u>Get rid of dependency of the coordinates (X, Y, Z) from each other.</u>**

$$x = r \cdot \cos(t)$$
$$y = r \cdot \sin(t)$$
$$z = h$$

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- Get rid of dependency of the coordinates (X, Y, Z) from each other.
- **Can be extended to higher objects. (4th parameter)**

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- **More DOF for controlling curves and surfaces.**

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- **<u>Transformations (distinct separation between shape and trans. info).</u>**

R=7 circle at 0,0	R=7 circle at 4,3
$x = 7 \cdot \cos(t)$ $y = 7 \cdot \sin(t)$	$x = 4 + 7 \cdot \cos(t)$ $y = 3 + 7 \cdot \sin(t)$
$x^2 + y^2 - 49 = 0$	$x^2 + y^2 - 8 \cdot x - 6 \cdot y - 24 = 0$

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- **Vector – matrix multiplication (not good for you, good for comp).**

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- **No problems for slope calculation.**

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- **Discretizing entities**

Parametric Representation (PR)

$$\begin{cases} X = f(t) \\ Y = g(t) \\ Z = h(t) \end{cases} \rightarrow \begin{aligned} x &= r \cdot \cos(t) \\ y &= r \cdot \sin(t) \\ z &= h \end{aligned} \rightarrow \begin{aligned} x &= r \cdot \cos(t) \\ y &= r \cdot \sin(t) \end{aligned}$$

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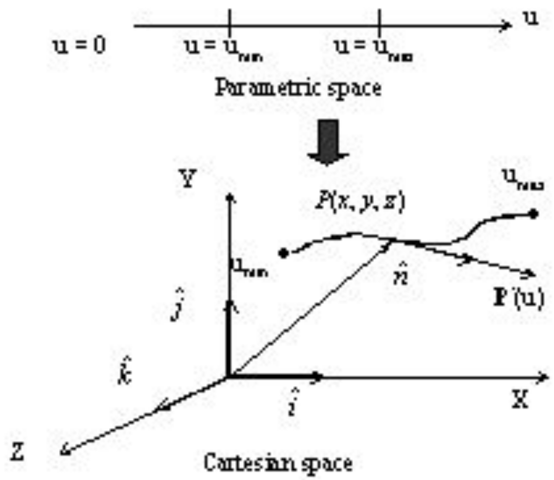
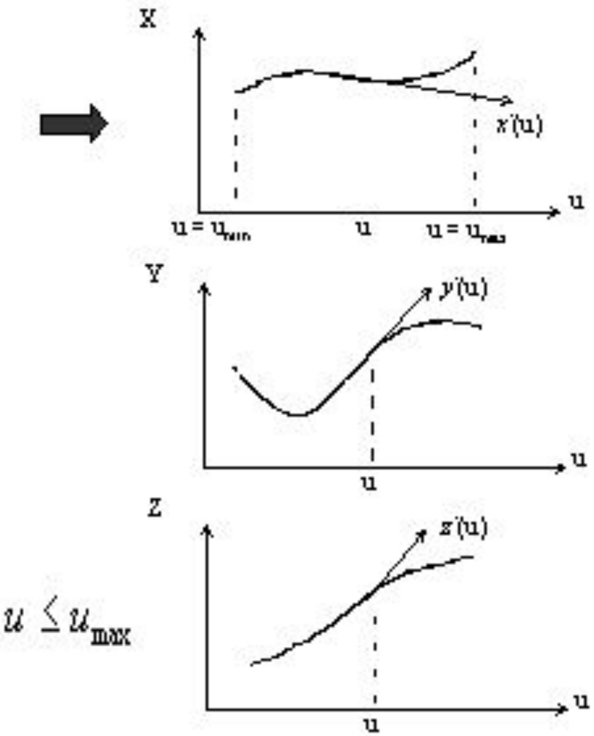
PR of 3D Curve





Curve components in parametric space

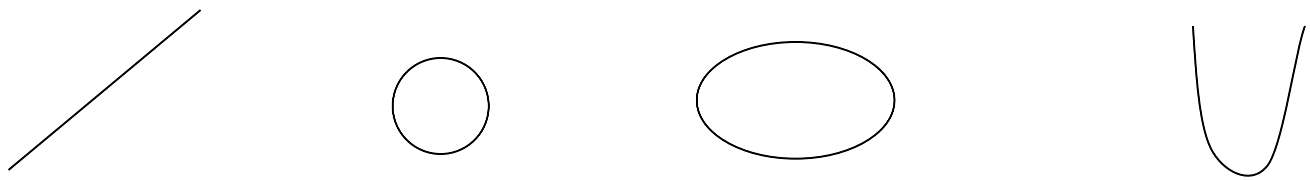
Tangent vector $\mathbf{P}'(u) = \dfrac{d\mathbf{P}(u)}{du}$

or

$$\mathbf{P}'(u) = \begin{bmatrix} x' & y' & z' \end{bmatrix}^T = \begin{bmatrix} x'(u) & y'(u) & z'(u) \end{bmatrix}^T, \; u_{min} \le u \le u_{max}$$

PR of Analytic Curves

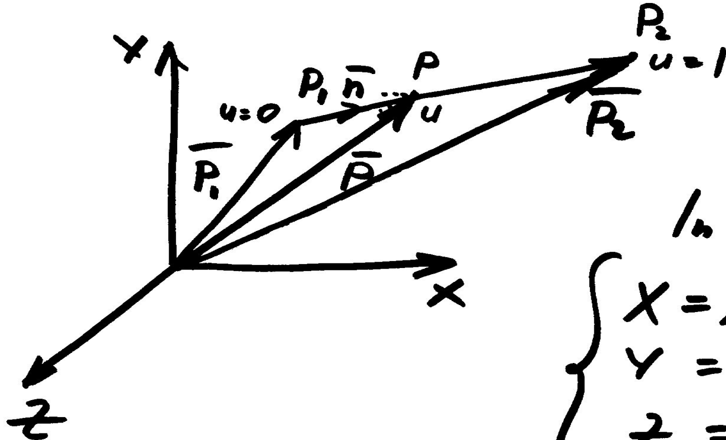
Analytic curves are defined by analytic equations



- Compact form for representation
- Simple computation of properties

- Little practical use
- No local control

Lines: 2 points



In vector form:

$$\bar{P} - \bar{P}_1 = u(\bar{P}_2 - \bar{P}_1)$$

$$\bar{P} = \bar{P}_1 + u(\bar{P}_2 - \bar{P}_1) \quad 0 \le u \le 1$$

In scalar form

$$\begin{cases} X = X_1 + u(X_2 - X_1) \\ Y = Y_1 + u(Y_2 - Y_1) \\ Z = Z_1 + u(Z_2 - Z_1) \end{cases}$$

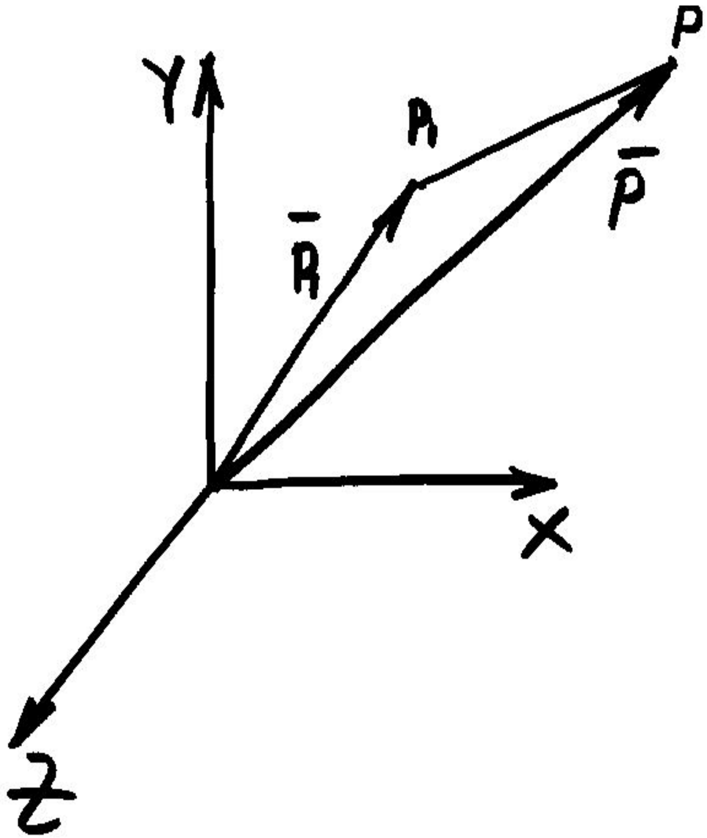
Tangent vector — $\bar{P}' = \bar{P}_2 - \bar{P}_1$

in scalar form $\begin{cases} x' = x_2 - x_1 \\ y' = y_2 - x_1 \\ z' = z_2 - z_1 \end{cases}$

no „u" $\longrightarrow$ constant slope

2 points in a line database

Lines: point and direction

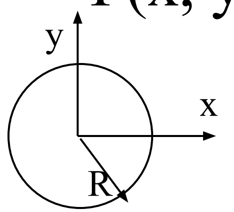


You know P_1; $\bar{n}$; L

$\bar{P} = \bar{P}_1 + L\bar{n}$ $0 \le L \le L_{max}$

Parametric equation from NP Implicit Equation: Example

F(x, y) = 0

For 

$x^2 + y^2 - R^2 = 0$

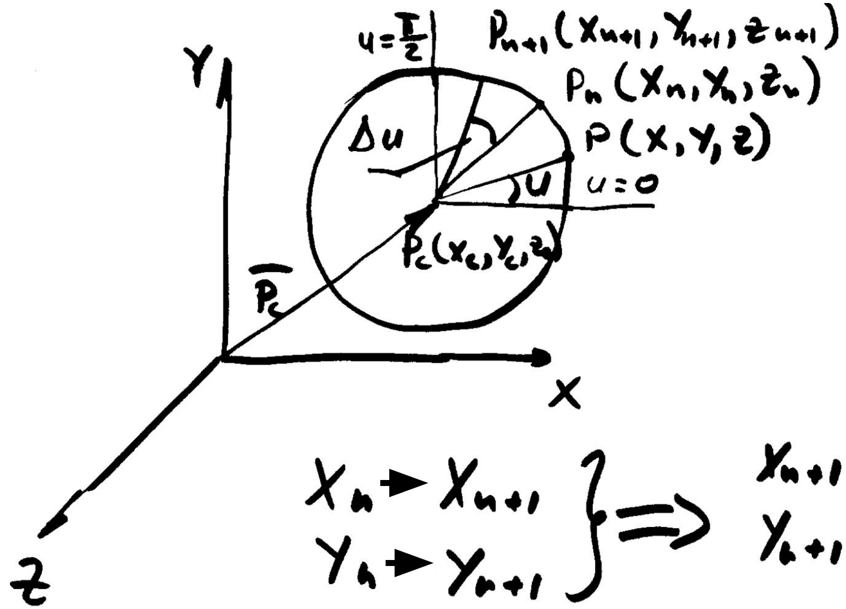
$$x = R\cos 2\pi u, \text{ where } 0 \le u \le 1$$

$$y = R\sin 2\pi u, \text{ where } 0 \le u \le 1$$

Parametric equation:

$$P(u) = [R\cos 2\pi u, R\sin 2\pi u]^T, \qquad 0 \le u \le 1$$

Circles



Basic param. equation:

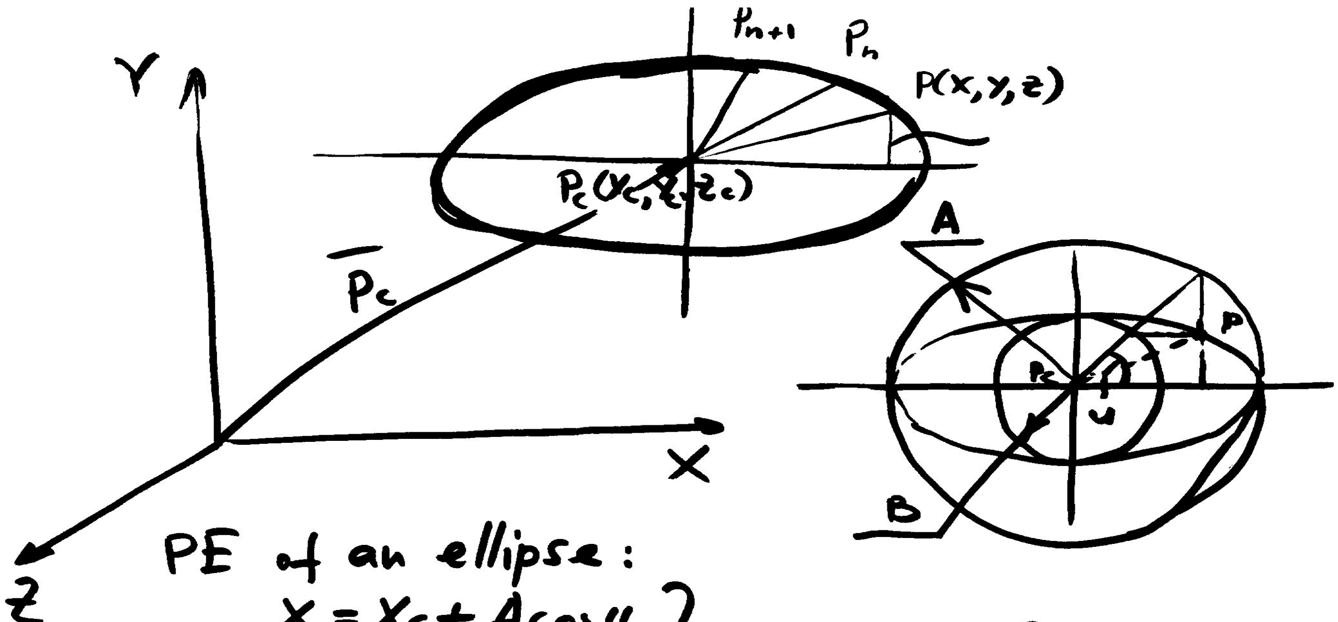
$$\left.\begin{aligned} x &= x_c + R\cos u \\ y &= y_c + R\sin u \\ z &= z_c \end{aligned}\right\} \quad 0 \le u \le 2\pi$$

$$\left.\begin{aligned} x_n &\rightarrow x_{n+1} \\ y_n &\rightarrow y_{n+1} \end{aligned}\right\} \Longrightarrow \begin{aligned} x_{n+1} &= x_c + (x_n - x_c)\cos\Delta u - (y_n - y_c)\sin\Delta u \\ y_{n+1} &= y_c + (y_n - y_c)\cos\Delta u + (x_n - x_c)\sin\Delta u \end{aligned}$$

For arcs $\left.\begin{aligned} x &= x_c + R\cos u \\ y &= y_c + R\cos u \\ z &= z_c \end{aligned}\right\} \quad u_s \le u \le u_e$

+ Direction of arc creation — CCW

Ellipses



PE of an ellipse:

$$\left.\begin{aligned} X &= X_c + A\cos u \\ Y &= Y_c + B\sin u \\ Z &= Z_c \end{aligned}\right\} \quad 0 \le u \le 2\pi$$

For computat. & display purposes:

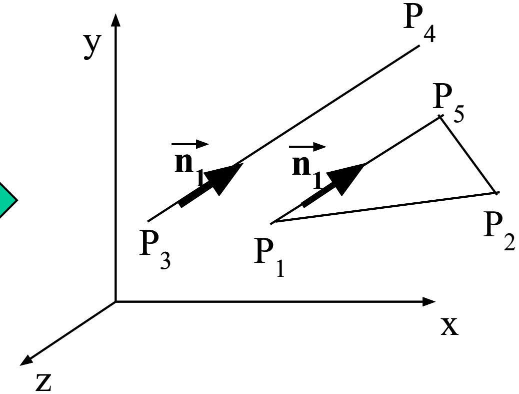
$$X_{n+1} = X_c + (X_n - X_c)\cos\Delta u - \frac{A}{B}(Y_n - Y_c)\sin\Delta u$$

$$Y_{n+1} = Y_c + (Y_n - Y_c)\cos\Delta u + B/A\,(X_n - X_c)\sin\Delta u$$

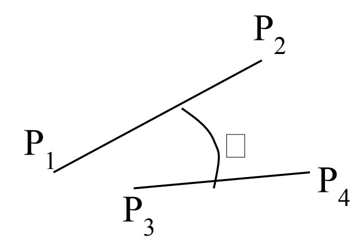
$$Z_{n+1} = Z_n$$

Examples

- Find the equation and endpoints of a line that passes through point P1, parallel to an existing line, and is trimmed by point P2.



- Relate the following CAD commands to their mathematical foundations:
  - The command to measure the angle between two intersecting lines.



  - The command to find the distance between a point and a line.