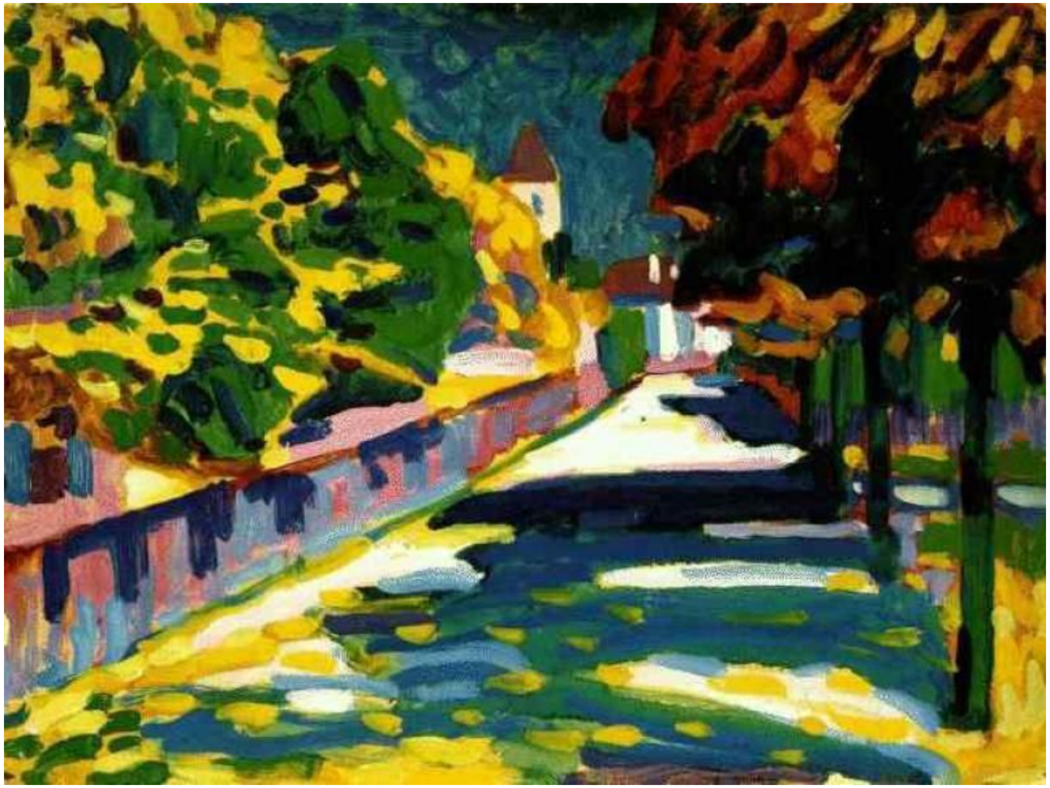


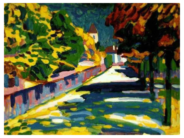
MOSCOW EXCHANGE

Moscow University
Risk Management

Class #7 – Derivatives Pricing II

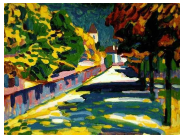
Lecturer: Luis A. B. G. Vicente

November/2015

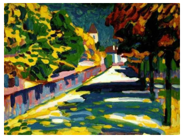


Class #7 – Derivatives Pricing II

1. The binomial model
2. The Black –Scholes model
3. Monte Carlo pricing
4. Annex



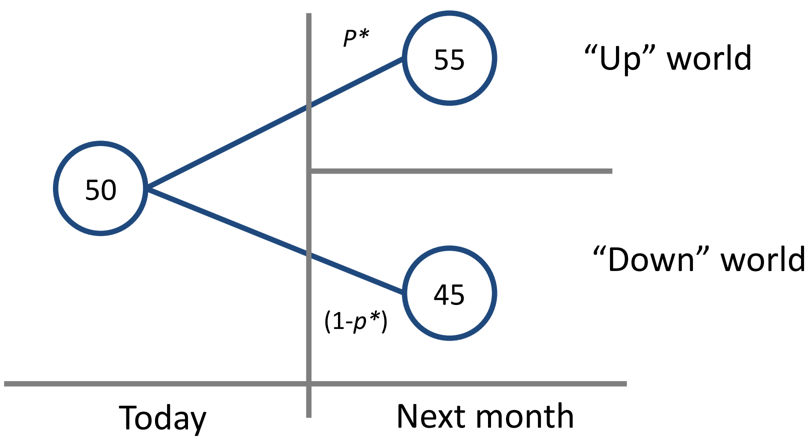
Class #7 – Derivatives Pricing II

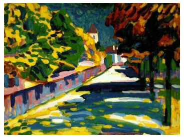


The binomial model

A risk neutral portfolio

- Suppose we have a two period model of price dynamics represented by t=0 (today) and t=M (Next month)
- Also consider that there are only two possible states of the world: "up" and "down"
- The probability of "up" occurring is p^*, and the probability of "down" occurring is $(1 - p^*)$
- For instance, consider a stock with price equal to S=50 on t=0 and S_{up}=55 on t=1 and S_{down}=45 on t=1



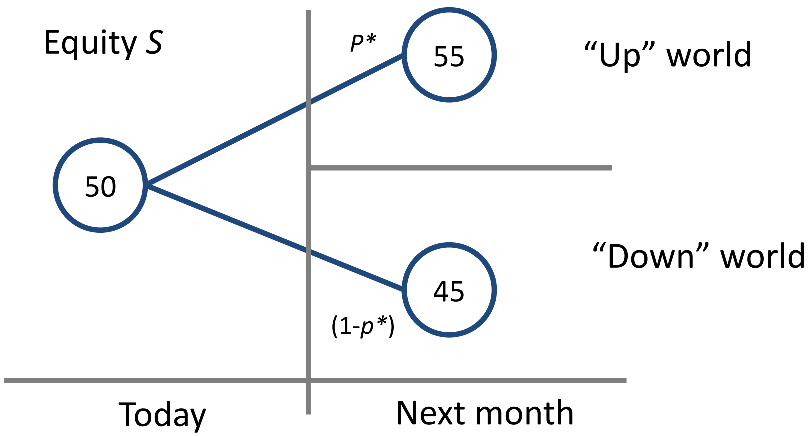


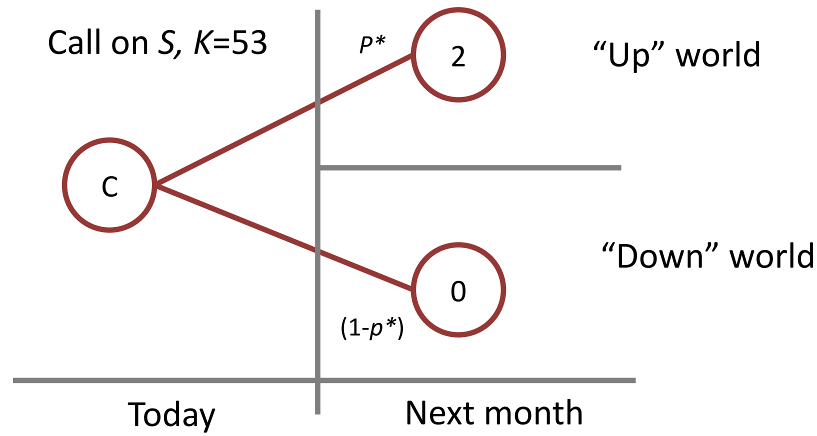
The binomial model

A risk neutral portfolio

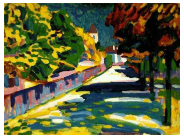
Now suppose we have a call option on *S* with strike price 53 that matures in one month

The option payoff in the “up” and “down” worlds would be:





How can we calculate the price *C* of the option today?



The binomial model

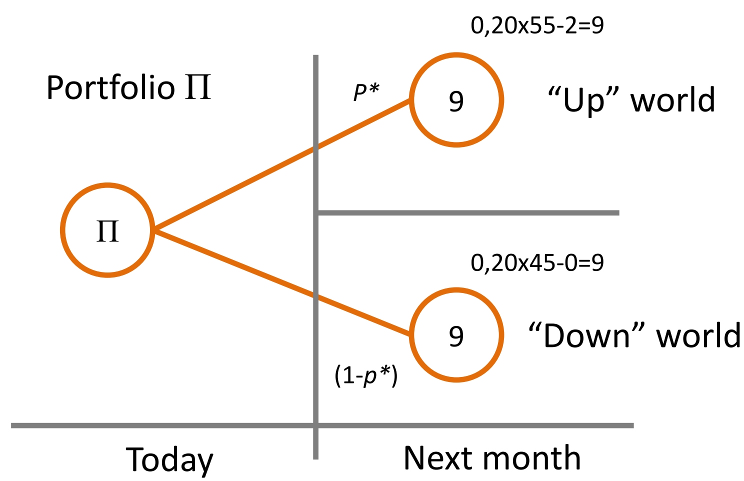
A risk neutral portfolio

Can we build a portfolio so we have a portfolio that has the same value no matter the future state of the world?

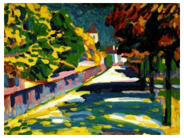
Suppose you have a portfolio that is short 1 call and long Δ stocks

So we want find a quantity Δ that:

$$\underbrace{\Delta \text{ x } 55 - 1 \text{ x } 2}_{\text{"Up" world}} = \underbrace{\Delta \text{ x } 45 - 1 \text{ x } 0}_{\text{"Down" world}} \Rightarrow \Delta = 0{,}20$$



Hence portfolio P has the same value in all states of the world



The binomial model

A risk neutral portfolio

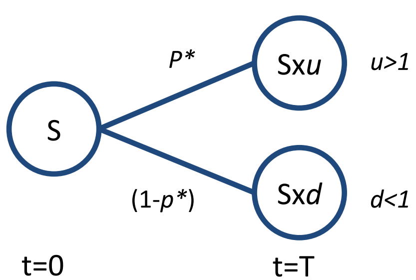
This is a riskless portfolio, as it has the same value no matter what happens to the world

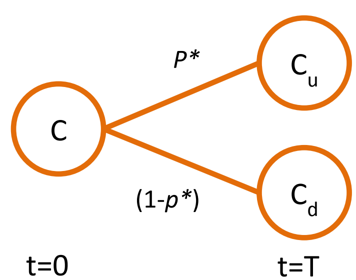
We know from non-arbitrage theory that riskless portfolios must earn the risk-free interest rate

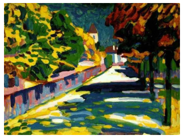
Suppose the annual interest rate is 20% cc, hence: $\Pi = 9 \times e^{-0.20 \times {}^{1}/_{12}}$ =8.85

We know that $\Pi = \Delta \times S - C$, therefore $C = 0.2 \times 50 - 8.85 = 1.15$

A general model

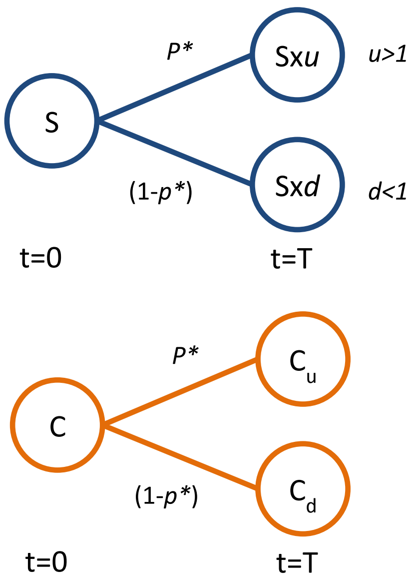






The binomial model

A general model



Algebraic solution

$$S \times u \times \Delta - C_u = S \times d \times \Delta - C_d$$

$$\Delta = \frac{C_u - C_d}{S \times u - S \times d}$$

The cost of setting up the portfolio Π has to be equal to the present value of its (riskless) payoff, hence

$$S \times \Delta - C = e^{-rT}(S \times u \times \Delta - C_u)$$

$$C = S \times \Delta - e^{-rT}\,(S \times u \times \Delta - C_u)$$

Making the appropriate substitutions, we have

$$C = e^{-rT}(p \times C_u + (1-p) \times C_u) \text{ with } p = \frac{e^{rT} - d}{u - d}$$

Hence, the price of the option can be viewed as the present value of its expected future price considering the probability measure *p*

$$C_{t=0} = e^{-rT} \times E[C_{t=T}]$$

Can we match p^* with p?

The binomial model

Risk neutral valuation

Risk neutral valuation assumes that people are indifferent to risk

Therefore, the expected return on all assets equals the risk-free interest rate

So, we can write $E[S_{t=T}] = S_{t=0} \times e^{r \times T}$

or simply $E[S_T] = S_0 e^{rT}$

In our case, we consider *p* to be risk neutral probabilities

The key variables of the model are then *u*, *d* and *r*.

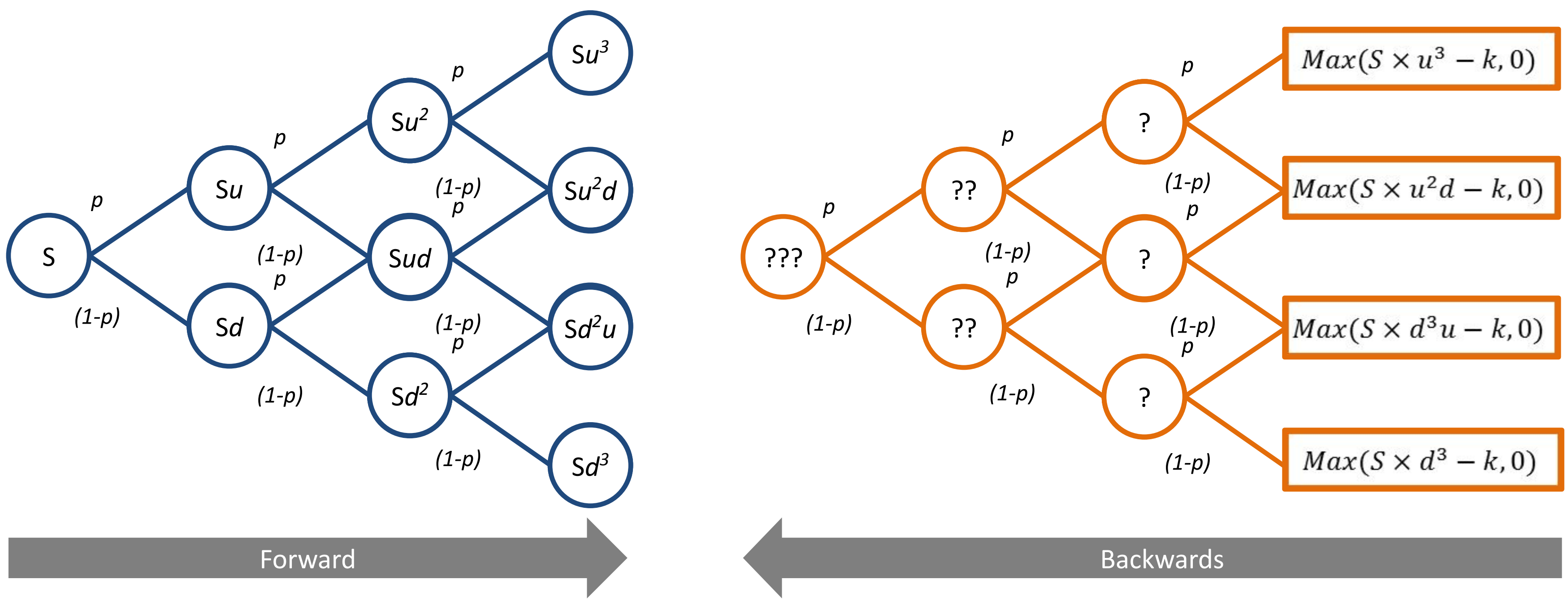
Making $u = e^{\sigma\sqrt{\Delta t}}$ and $e^{-\sigma\sqrt{\Delta t}}$ allows us to match the volatility of the underlying *S* and is consistent with RNM

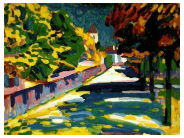
The binomial model

Pricing options with binomial trees

It is not difficult to notice that we can build recursive structures based on the binomial model

Hence, we can price options considering multiple periods, by working backwards.





The binomial model

Pricing options with binomial trees

Consider, for instance, pricing a call option with strike price 105 that matures in 5 weeks

The current spot price is 100, interest rate is 10% and the volatility of the underlying is 20%

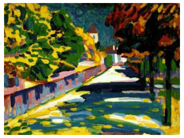
Hence, the binomial trees for the underlying asset and for the option are:

Now	Week 1	Week 2	Week 3	Week 4	Week 5
					114.88
				111.73	
			108.68		108.68
		105.70		105.70	
	102.81		102.81		102.81
100.00		100.00		100.00	
	97.26		97.26		97.26
		94.60		94.60	
			92.02		92.02
				89.50	
					87.05

Forward

Now	Week 1	Week 2	Week 3	Week 4	Week 5
					9.88
				6.93	
			4.57		3.68
		2.89		1.94	
	1.77		1.02		0.00
1.07		0.54		0.00	
	0.28		0.00		0.00
		0.00		0.00	
			0.00		0.00
				0.00	
					0.00

Backwards



The binomial model

Pricing options with binomial trees

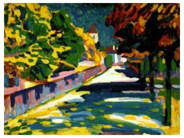
Pricing a similar put option with strike price 110 can be equally easy

Now	Week 1	Week 2	Week 3	Week 4	Week 5
					114.88
				111.73	
			108.68		108.68
		105.70		105.70	
	102.81		102.81		102.81
100.00		100.00		100.00	
	97.26		97.26		97.26
		94.60		94.60	
			92.02		92.02
				89.50	
					87.05

Forward

Now	Week 1	Week 2	Week 3	Week 4	Week 5
					0.00
				0.62	
			2.25		1.32
		4.38		4.08	
	6.72		6.77		7.19
9.15		9.37		9.79	
	11.89		12.31		12.74
		14.76		15.18	
			17.56		17.98
				20.29	
					22.95

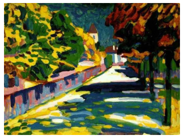
Backwards



The binomial model

Pricing options with binomial trees – some stylized facts

- Very flexible and intuitive approach
- Can be computationally intensive
- Values converge as $\Delta t \rightarrow 0$
- The model assumes that you are rebalancing your risk-free portfolio at every Δt
- Distribution of future prices: binomial$\rightarrow$lognormal



Class #7 – Derivatives Pricing II

The Black Scholes model

The Black Scholes model

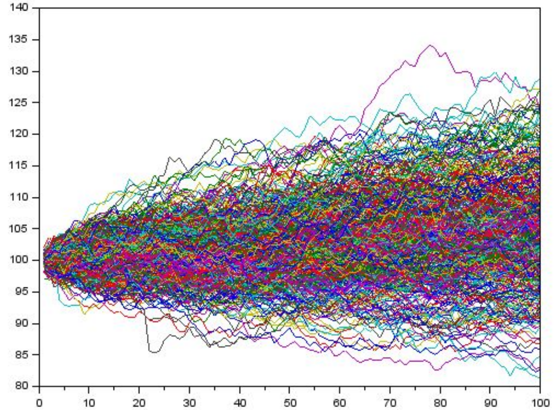
The Black Scholes model is a continuous time model, so we work in terms of *dt*, not Δt.

Asset prices are supposed to conform with the following stochastic process:

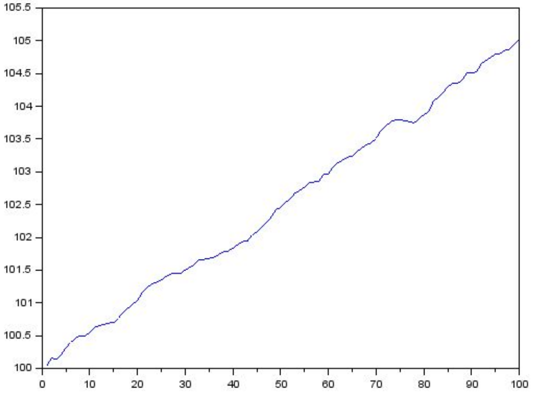
$$\frac{dS}{S} = \mu dt + \sigma dX \equiv dS = S\mu dt + S\sigma dX \text{ and } X \sim N(0,1)$$

Geometric brownian motion

Simulated process



Average



MOSCOW EXCHANGE

The Black Scholes model

The Black Scholes model

If we have a derivative (contingent instrument) *f* based on *S*, then, according to Ito's lemma:

$$df = \left(\frac{df}{dt} + \frac{df}{dS}\mu S + \frac{1}{2}\frac{d^2 f}{dS^2}\sigma^2 S^2\right)dt + \frac{df}{dS}\sigma S dX$$

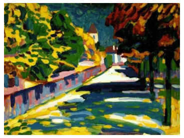
We then build a portfolio Π with $\frac{df}{dS}$ shares and -1 derivative, we have $\Delta\Pi = \left(-\frac{df}{dt} - \frac{1}{2}\frac{d^2 f}{dS^2}\sigma^2 S^2\right)\Delta t$

Because thus portfolio does not contain *dX*, it should be riskless, or in other words, yield the risk-free rate, hence

$$\left(-\frac{df}{dt} - \frac{1}{2}\frac{d^2 f}{dS^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{df}{dS}S\right)\Delta t$$

$$\frac{df}{dt} + rS\frac{df}{dS}S + \frac{1}{2}\frac{d^2 f}{dS^2}\sigma^2 S^2 = rf$$

The Black Scholes PDE



The Black Scholes model

The Black Scholes model

It can be shown that the closed-formula solution for the BSPDE for a call option is

$$c = S \times N(d1) - K \times e^{-rt} \times N(d2)$$

Also, the closed-formula solution for the BSPDE for a put option is

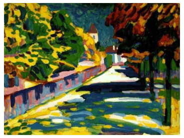
$$p = K \times e^{-rt} \times N(-d2) - S \times N(-d1)$$

Where

$$d1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + {\sigma^2}/{2}\right)t}{\sigma\sqrt{t}}$$

$$d2 = \frac{ln\left(\frac{S}{K}\right) + \left(r - {\sigma^2}/{2}\right)t}{\sigma\sqrt{t}} = d1 - \sigma\sqrt{t}$$

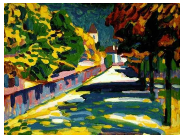
N(.) is the normal cumulative distribution



The Black Scholes model

The Black Scholes model

- Consider our previous example of pricing a call option with strike price 105 that matures in 5 weeks
- Again, the current spot price is 100, interest rate is 10% and the volatility of the underlying is 20%
- In this case the price is 1.01 (compared to 1.07 from the binomial model)
- In the put example (K=110) the price is 9.19 (compared to 9.15 from the binomial model)
- Key difference: continuous time vs. discrete time



The Black Scholes model

Hedging and the Greeks

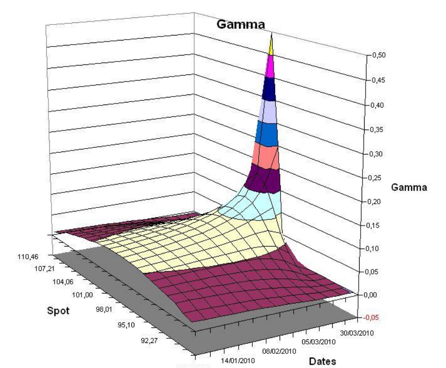
Delta $\Delta = \frac{\partial V}{\partial S}$ is the sensitivity to asset price.

Gamma $\Gamma = \frac{\partial^2 V}{\partial^2 S}$ is the sensitivity of delta to S

Theta $\Theta = \frac{\partial V}{\partial t}$ is time rate of change of V

Vega $\nu = \frac{\partial V}{\partial \sigma}$ is the sensitivity of V to sigma (volatility)

Rho $\rho = \frac{\partial V}{\partial r}$ is the sensitivity of V to the interest rate

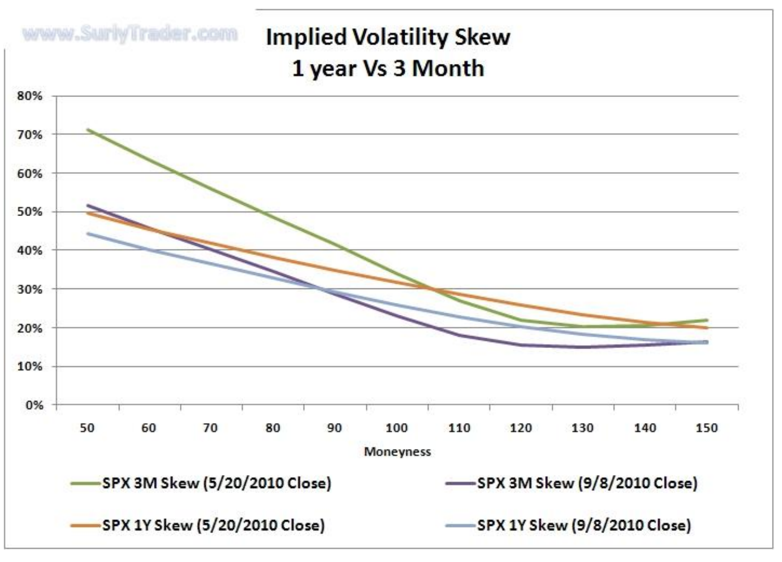


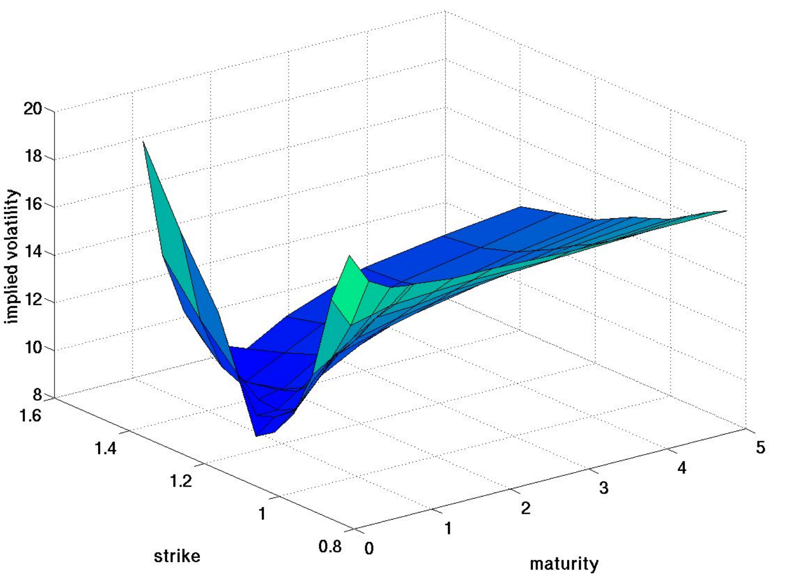
MOSCOW EXCHANGE

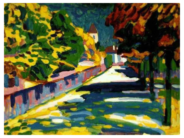
Black Scholes model

Volatility Surface

- Volatility is not constant as in the BS model
- Traders price options by the implied volatility seen in the market
- Implied volatility is “the wrong number to put in the wrong (Black) formula to get the right price” R. Rebonato







The Black Scholes model

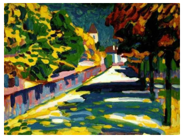
Dynamic Delta hedging

Delta $\Delta = \frac{\partial V}{\partial S}$ is the sensitivity to asset price ; if we set up $\Pi = V - \Delta S$ the portfolio is riskless ,in theory

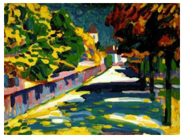
“In theory there is no difference between theory and practice. In practice there is.” Yogi Berra

Factors to consider : time to maturity , level of at the moneyness, volatility , interest rates, dividends, liquidity pockets

In reality, the trader needs constantly rebalancing the delta over time to keep the portfolio “risk neutral”



Class #7 – Derivatives Pricing II

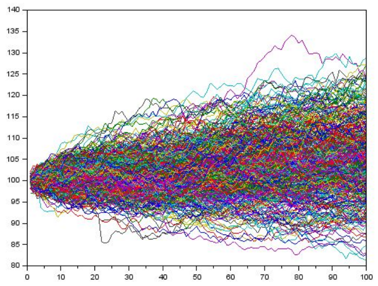


Monte Carlo pricing

Monte Carlo is a simulation technique

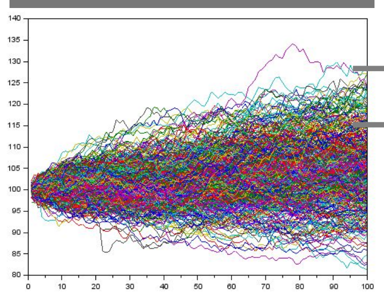
We simulate every possible path for the security price based on a pre-defined stochastic process

Simulated process



The price of a contingent claim is given by present value of the average of the potential payoffs

Simulated process



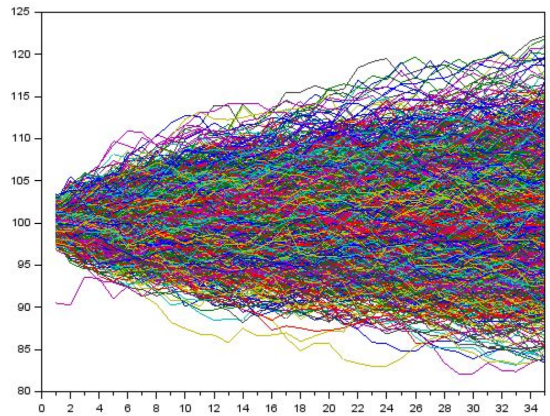
Payoff>0

Monte Carlo pricing

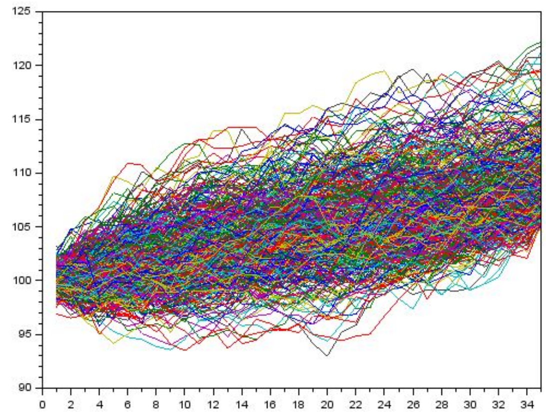
Monte Carlo is a simulation technique

Pricing our example (call option with strike price 105 that matures in 5 weeks) using 2000 simulations and 35 steps

Simulated process (2000 paths)

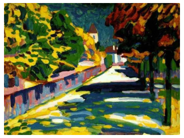


S>K at T (510 paths)



In this case the price of the call is 1.00 (compared to 1.01 from BS and 1.07 from the binomial model)

In the put example the price is 9.12 (compared to 9.19 from BS and 9.14 from the binomial model)



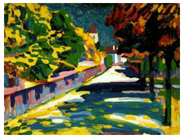
Monte Carlo pricing

MC main advantage

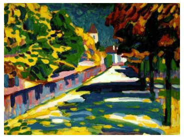
- It’s a very flexible approach, can calculate prices for exotic products

Drawbacks with MC

- Slow convergence
- Computationally expensive if multi -period or non- recombinant
- Stability of the Greeks



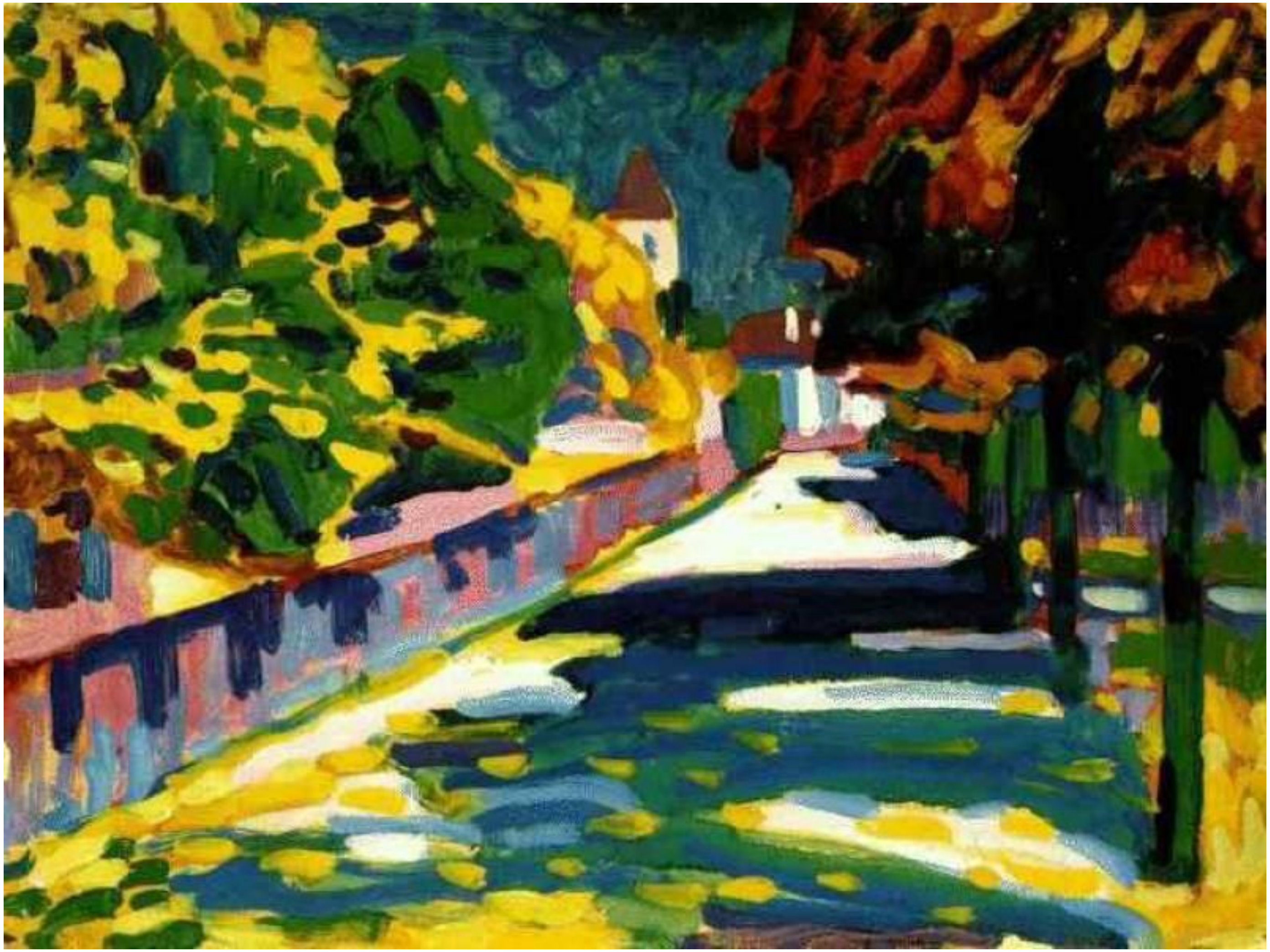
Class #7 – Derivatives Pricing II



Annex

Useful References

- Options, Futures and Other Derivatives, John Hull, (2014);
- The Mathematics of Financial Derivatives , Paul Wilmott, Sam Howison, Jeff Dewynne (1995);
- Dynamic Hedging, Nassim Taleb(1997);



Vassily Vassilyevich Kandinsky, *Autumn in Bavaria*, 1908