NUFYP Mathematics

4.4 Binomial expansions

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Lecture Outline

- Expansion of
  - positive integer
    - Finite series
  - any other rational number
    - Infinite series
      Needs a convergence condition

Brief History

Euclid
4^{th} century BC
Found the formula for $(a+b)^2$.

Blaise Pascal
17^{th} century
Gave the form that we will learn today for a positive integral index.

James Gregory and Isaac Newton
17^{th} century
Worked on rational indices.
Convergence was not considered.

Carl Friedrich Gauss
19^{th} century
Proved the convergence condition for rational indices.

Applications

- Binomial expansions are used in probability for predicting nation's economy, weather forecasting, etc.

  (We will further study this in the Spring semester)

- Binomial expansions are also used in computer science, e.g. distribution of IP addresses

Pascal's triangle

The numbers you saw in the preview activity form a triangle called Pascal's triangle.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Did you find the pattern and the next two rows?

1

1 1

1 + 2 1

1 3 + 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Binomial expansions using Pascal's triangle

- Each row of Pascal's triangle gives the coefficients of $(a+b)^n$ for nonnegative integers, $n = 0,1,2,\ldots$

$$
\begin{array}{lccccccccc}
(a+b)^0 = & & & & & 1 & & & & \\
(a+b)^1 = & & & & 1a & + & 1b & & & \\
(a+b)^2 = & & & 1a^2 & + & 2ab & + & 1b^2 & & \\
(a+b)^3 = & & 1a^3 & + & 3a^2b & + & 3ab^2 & + & 1b^3 & \\
(a+b)^4 = & 1a^4 & + & 4a^3b & + & 6a^2b^2 & + & 4ab^3 & + & 1b^4
\end{array}
$$

Example 1

- Use Pascal's triangle to find the expansion of $(x+2y)^3$.

Solution

$$(x+2y)^3 = 1x^3 + 3x^2(2y)^1 + 3x(2y)^2 + 1(2y)^3$$

These coefficients came from the 4th row of Pascal's triangle.

$$= x^3 + 6x^2y + 12xy^2 + 8y^3$$

Your turn!

- Use Pascal's triangle to find the expansion of $(x-2y)^4$.

Solution

$$(x-2y)^4$$

$$= 1x^4 + 4x^3(-2y)^1 + 6x^2(-2y)^2 + 4x(-2y)^3 + 1(-2y)^4$$

$$= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$$

Binomial expansions using combinations

- Can you think of any drawback of using Pascal's triangle?

Yes, it is time consuming to find coefficients for large n.
Combinations provide a faster method.

Combinations

- In mathematics, a combination is a selection of items from a collection where the order of selection does not matter.

  e.g. From 26 alphabets, selecting (a,b) is identical to selecting (b,a)

- In contrast to a combination, the order of selection does matter for a permutation. That is, (a,b) and (b,a) are distinct.

- We will learn permutations and more combinations in semester 2.

Combinations

${}^{n}C_r$ or $\binom{n}{r}$ is the number of all possible combinations of selecting r items from a group of n items

$$ {}^{n}C_r = \frac{n!}{r!(n-r)!} $$

where $n! = n \times (n-1) \times \cdots \times 2 \times 1$

Note: $0! = 1$ $\qquad$ $(-3)! = \text{Undefined}$

Combinations

Remember the following identities.

- ${}^{n}C_0 = 1$
- ${}^{n}C_1 = n$
- ${}^{n}C_r = {}^{n}C_{n-r}$

(From a group of n items, selecting r items is equivalent to selecting $n-r$ items)

Try proving these identities by using the formula ${}^{n}C_r = \frac{n!}{r!(n-r)!}$

Alternative formula for ${}^{n}C_r$

- $${}^{n}C_r = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\cdots(n-r+1)\cancel{(n-r)!}}{r!\,\cancel{(n-r)!}}$$

$$= \frac{n(n-1)\cdots(n-r+1)}{r!}$$

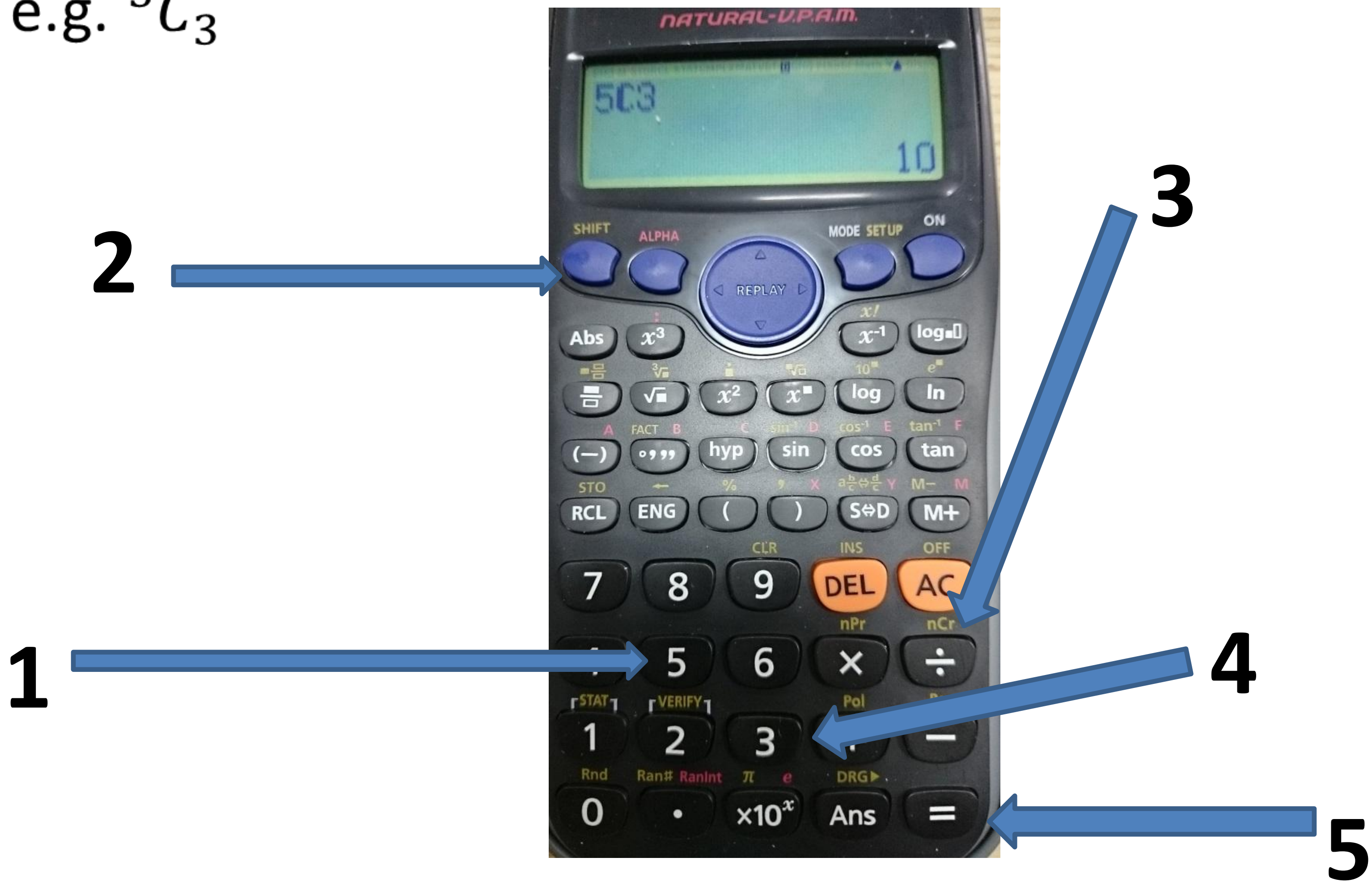
In the numerator, there are r numbers.

The formula may look more complicated, but you may find it easier in actual computation.

e.g. $${}^{n}C_3 = \frac{n(n-1)(n-2)}{3!}$$

Using the calculator to find $^{n}C_{r}$

e.g. $^{5}C_{3}$



Example 2

From the letters ABC, how many different ways of choosing 2 letters?

Solution

Using the formula, ${}^{3}C_{2} = \frac{3!}{2!(3-2)!} = 3$

We can verify the answer by counting all possible combinations; AB, AC, BC

Computations of some ^nC_r

- $^0C_0 = 1$
- $^1C_0 = {}^1C_1 = 1$
- $^2C_0 = 1, \quad {}^2C_1 = 2, \quad {}^2C_2 = 1$
- $^3C_0 = 1, \quad {}^3C_1 = 3, \quad {}^3C_2 = {}^3C_1 = 3, \quad {}^3C_3 = 1$
- $^4C_0 = 1, \quad {}^4C_1 = 4, \quad {}^4C_2 = \frac{4\times 3}{2!} = 6$
  $^4C_3 = {}^4C_1 = 4, \quad {}^4C_4 = 1$

The numbers look familiar!
These numbers are the numbers that we saw in the Pascal’s triangle.

Comparisons: Pascal’s triangle and Combinations

$$^{0}C_{0}$$

$$^{1}C_{0} \qquad ^{1}C_{1}$$

$$^{2}C_{0} \qquad ^{2}C_{1} \qquad ^{2}C_{2}$$

$$^{3}C_{0} \qquad ^{3}C_{1} \qquad ^{3}C_{2} \qquad ^{3}C_{3}$$

$$^{4}C_{0} \qquad ^{4}C_{1} \qquad ^{4}C_{2} \qquad ^{4}C_{3} \qquad ^{4}C_{4}$$

This allows us to find the coefficients without drawing the Pascal’s triangle. We can easily find a particular term in the expansion.

Binomial expansions using combinations

- Using the previous idea, we obtain the following formula for binomial expansions.

$$\boxed{(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n b^n}$$

where n is a positive integer.

Alternatively,

$$\boxed{(a+b)^n = {}^nC_0 b^n + {}^nC_1 a b^{n-1} + {}^nC_2 a^2 b^{n-2} + \cdots + {}^nC_n a^n}$$

Example 3

- Find the coefficient of x^6y^2 in the expansion of $(x+y)^8$.

Solution

$n = 8$ and by looking at the indices for x and y,

$$^8C_6 = {}^8C_2 = \frac{8 \times 7}{2!} = 28$$

Your turn!

- Find the coefficient of x^3y^{98} in the expansion of $(x+y)^{101}$.

Solution

$n = 101$ and by looking at the indices for x and y,

$$^{101}C_3 = {}^{101}C_{98} = \frac{101 \times 100 \times 99}{3!} = 166650$$

Example 4

- Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{2x}\right)^3$.

The power 3 is small enough for you to find the entire expansion, but in this example, we will focus on how to find the term *without* finding the whole expansion.

Solution

- The term independent of x is a constant term.

$$ {}^{3}C_{r}(x^{2})^{r}\left(-\frac{1}{2x}\right)^{3-r} $$

$$ 2r = 3 - r \quad \rightarrow \quad r = 1 $$

$$ {}^{3}C_{1}(x^{2})^{1}\left(-\frac{1}{2x}\right)^{3-1} = 3x^{2}\frac{1}{4x^{2}} = \frac{3}{4} $$

Your turn!

- Find the term independent of x in the expansion of $\left(3x + \frac{1}{3x^2}\right)^9$.

Solution

The term independent of x is a constant term.

$$ {}^{9}C_{r}(3x)^{r}\left(\frac{1}{3x^{2}}\right)^{9-r} $$

$$ r = 18 - 2r \quad \rightarrow \quad r = 6 $$

$$ {}^{9}C_{6}(3x)^{6}\left(\frac{1}{3x^{2}}\right)^{3} = \frac{9\times8\times7}{3!}\,(3^{6}x^{6})\left(\frac{1}{3^{3}x^{6}}\right) = 2268 $$

Binomial expansions when n is any rational number

- So far, we considered the expansion of $(a+b)^n$ where n is a positive integer.

What if n is a negative integer or a fraction?

We can generalize the formula

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \cdots + {}^nC_n b^n$$

to the case when n is a negative integer or a fraction.

Binomial expansions when n is any rational number

- There are important differences between the two cases (1) n is a positive integer and (2) n is a negative integer or a fraction

- The expansion of (1) is finite whereas that of (2) is infinite
- The expansion of (1) is always true whereas the expansion of (2) is valid only under a certain condition

Formula for the Expansion of $(1+x)^n$

- $(1+x)^n =$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots \qquad \text{for } |x| < 1$$

The infinite series on the right-hand side converges to $(1+x)^n$
if $|x| < 1$.
If not, we cannot expand $(1+x)^n$ as above.

Why $|x| < 1$?

- The proof of the previous expansion is beyond our course. However, we will look into a special case when $n = -1$.

Using the expansion as in the previous slide,

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

On the right, we see an infinite geometric series whose common ratio is $-x$, which converges to $\frac{1}{1+x}$ only if $|x| < 1$.

What is the importance of the expansion when n is a negative or fraction?

- By using this expansion, we can approximate a non-polynomial function by polynomials.

- Polynomials are the easiest function that we can handle. In many applications, polynomial approximations are used to analyze a complicated function.

Example 5

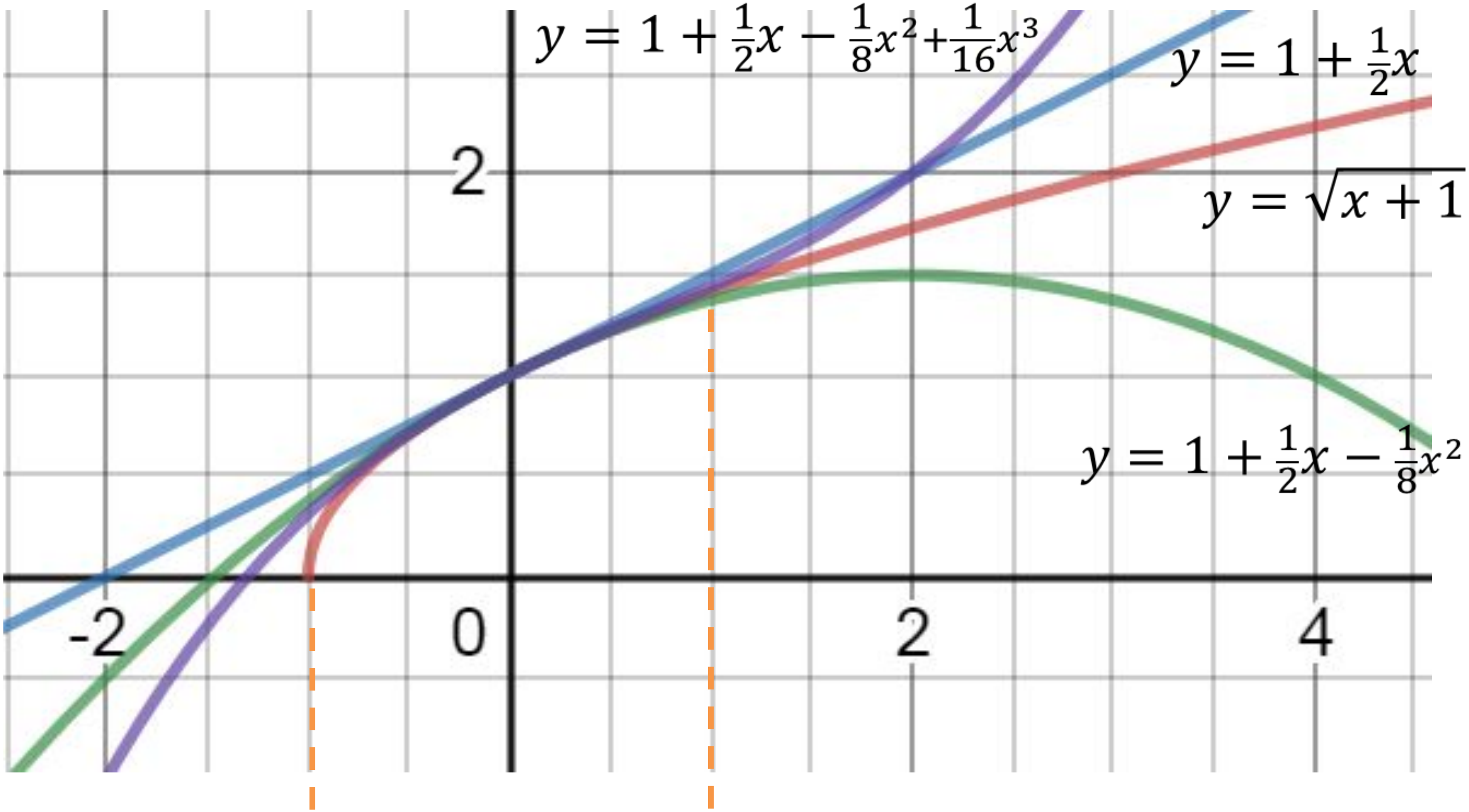
- Find the expansion of $\sqrt{1+x}$ up to and including the term in x^3. State the range of values of x for which the expansion is valid.

Solution

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots \qquad \text{when } |x| < 1$$

Solution (continued)



The expansion is valid for $|x| < 1$

Your turn!

- Find the expansion of $\frac{1}{(1+4x)^2}$ up to and including the term in x^3. State the range of values of x for which the expansion is valid.

Solution

- $(1 + 4x)^{-2}$

$$= 1 - 2(4x) + \frac{(-2)(-3)}{2!}(4x)^2 + \frac{(-2)(-3)(-4)}{3!}(4x)^3 + \cdots$$

$$= 1 - 8x + 48x^2 - 256x^3 + \cdots \quad \text{for } |4x| < 1 \rightarrow |x| < \frac{1}{4}$$

Expansion of $(\boldsymbol{a}+\boldsymbol{bx})^{\boldsymbol{n}}$

- When the first term is not 1, we need to factor out a, and the rest is the same as before.

- $(a+bx)^n = a^n\left(1+\frac{b}{a}x\right)^n =$

$$a^n\left(1+n\left(\frac{b}{a}x\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}x\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{b}{a}x\right)^3+\cdots\right)$$

for $\left|\frac{b}{a}x\right| < 1$.

Example 6

Find the first four terms in the binomial expansion of

$$\sqrt{4+x}$$

State the range of values of x for which the expansion is valid.

Solution

$$\sqrt{4+x} = (4+x)^{1/2} = 4^{1/2}\left(1+\frac{x}{4}\right)^{1/2}$$

$$= 2\left(1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{4}\right)^3+\cdots\right)$$

$$= 2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}+\cdots \text{ for } \left|\frac{x}{4}\right| < 1 \;\rightarrow\; |x| < 4$$

Your turn!

Find the first four terms in the binomial expansion of

$$\frac{1}{(2+3x)^2}$$

State the range of values of x for which the expansion is valid.

Solution

- $$(2+3x)^{-2} = 2^{-2}\left(1+\frac{3}{2}x\right)^{-2}$$

$$= \frac{1}{4}\left(1 - 2\left(\frac{3}{2}x\right) + \frac{(-2)(-3)}{2!}\left(\frac{3}{2}x\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{3}{2}x\right)^3 + \cdots\right)$$

$$= \frac{1}{4} - \frac{3}{4}x + \frac{27}{16}x^2 - \frac{27}{8}x^3 + \cdots \text{ for } \left|\frac{3}{2}x\right| < 1 \;\rightarrow |x| < \frac{2}{3}$$

Your turn!

- The expansion of $(a+bx)^{-2}$ may be approximated by $\frac{1}{4}+\frac{1}{4}x+cx^2$. Find the values of the constants a, b and c. State the range of values of x for which the expansion is valid.

Solution

- $$(a+bx)^{-2} = a^{-2}\left(1+\frac{b}{a}x\right)^{-2}$$

$$= \frac{1}{a^2}\left(1 - 2\left(\frac{b}{a}x\right) + \frac{(-2)(-3)}{2!}\left(\frac{b}{a}x\right)^2 + \cdots\right)$$

$$= \frac{1}{a^2} - \frac{2b}{a^3}x + \frac{3b^2}{a^4}x^2 + \cdots = \frac{1}{4} + \frac{1}{4}x + cx^2 + \cdots$$

$$\begin{cases} a = 2, b = -1, c = \frac{3}{16} \\ a = -2, b = 1, c = \frac{3}{16} \end{cases}$$

The expansion is valid when $\left|\frac{b}{a}x\right| < 1 \to |x| < 2$

Example 7

- Using the expansion from Example 5,

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

find the fraction that is an approximation to $\sqrt{5}$.

Solution

- Remember that the expansion is valid for $|x| < 1$, hence, we cannot substitute $x = 4$. Instead,

$$\sqrt{5} = \sqrt{4+1} = (4+1)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{1}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{1}{4}\right)^{\frac{1}{2}}$$

In the expansion of $\sqrt{1+x}$, we substitute $x = \frac{1}{4}$.

$$\sqrt{5} = 2\sqrt{1+\frac{1}{4}} = 2\left(1+\frac{1}{2}\left(\frac{1}{4}\right)-\frac{1}{8}\left(\frac{1}{4}\right)^2+\frac{1}{16}\left(\frac{1}{4}\right)^3+\cdots\right)$$

$$= 2+\frac{1}{4}-\frac{1}{64}+\frac{1}{512} = \frac{1145}{512}$$

Learning outcomes

4.4.1. Expand binomial expressions

4.4.2. Find a particular term in binomial expansions

4.4.3. Use a binomial expansion to approximate a certain function by a polynomial function

4.4.4. Find an estimate of a certain value using a polynomial approximation

Formulae

- Binomial expansion for a positive integer n

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n b^n$$

- Binomial expansion for any rational number n other than positive integers

$$(1+x)^n$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots$$

for $|x| < 1$

Preview activity: Mathematics of Finance

1. Using the binomial expansions, list the numbers in an ascending order.

$$A = \left(1 + \frac{r}{2}\right)^2, \qquad B = \left(1 + \frac{r}{4}\right)^4, \qquad C = \left(1 + \frac{r}{6}\right)^6,$$

where $r > 0$.

Can you generalize what you found?

2. Find the coefficients a_1, a_2, a_3 in terms of n where $n > 0$ and $r > 0$.

$$\left(1 + \frac{r}{n}\right)^n = 1 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots$$