NUFYP Mathematics

3.4 Trigonometry 4

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Lecture Outline

The addition and subtraction formulas $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$

Product-to-sum formulas

Sum-to-product formulas

Double angle formulas

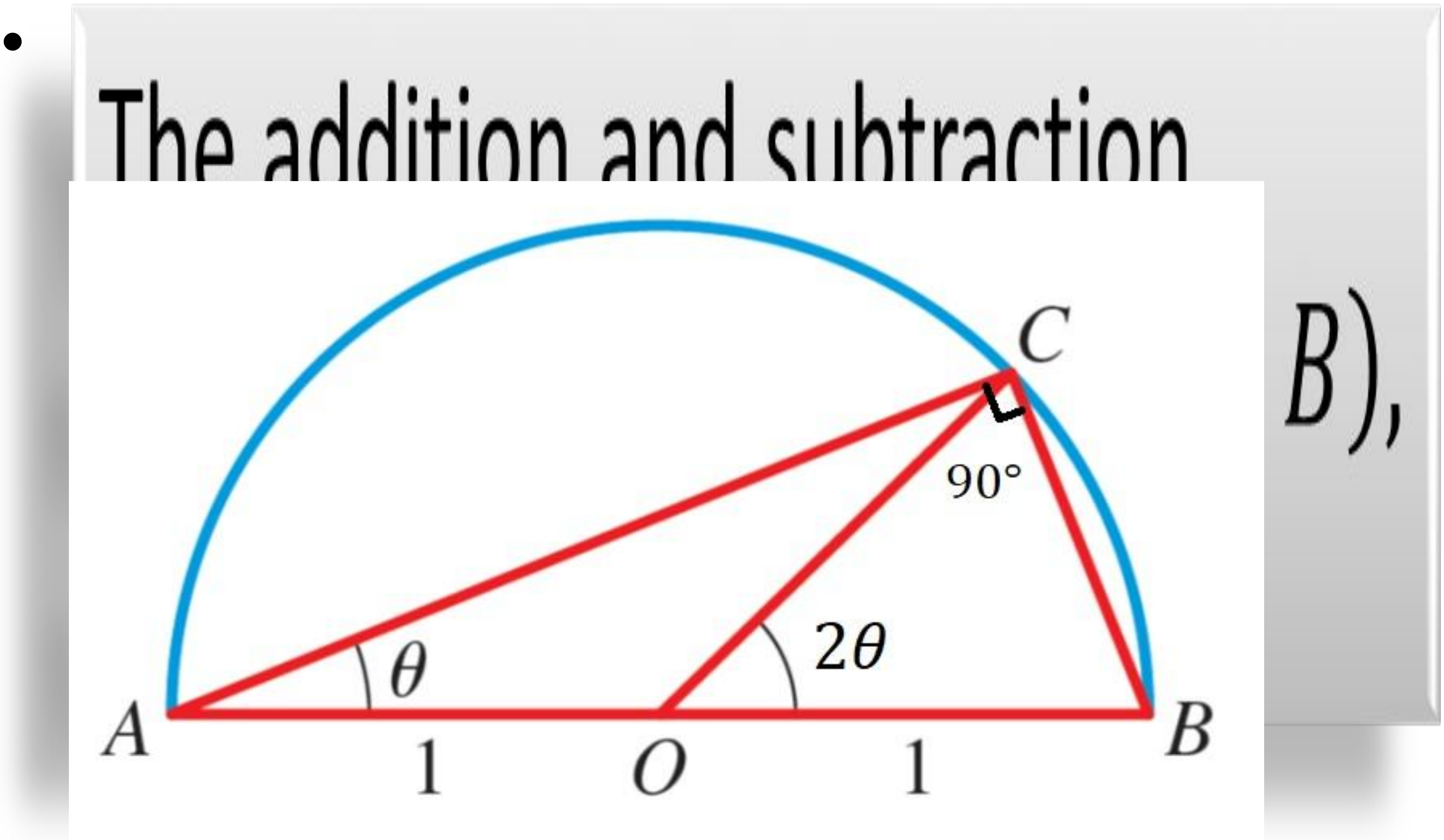
$\sin 2A$
$\cos 2A$
$\tan 2A$

R – formula:
$a\cos x + b\sin x = R\sin(x+\alpha)$
$R\cos(x-\alpha)$

Formulas for lowering powers

Half-angle formulas

Preview activity: 3.4 Trigonometry 4

- 

Preview activity 3.4. Solution

Area of the triangle ΔABC can be expressed in 2 ways:

1. $A_{\Delta ABC} = \frac{1}{2} \times AC \times CB,$

$AC = 2\cos\theta,\ CB = 2\sin\theta \ \rightarrow$

$A_{\Delta ABC} = 2\cos\theta\sin\theta$

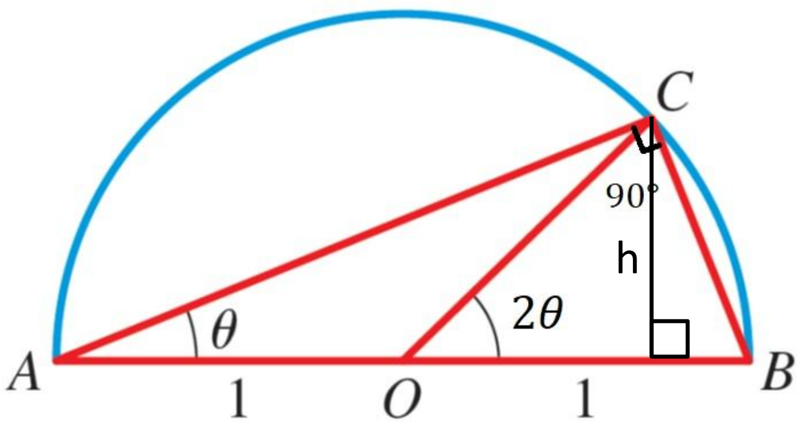
2. $A_{\Delta ABC} = \frac{1}{2} \times AB \times h$

$AB = 2, h = OC \times \sin 2\theta =$

$= 1 \times \sin 2\theta = \sin 2\theta$

$\rightarrow A_{\Delta ABC} = \sin 2\theta$

3. $\sin 2\theta = 2\cos\theta\sin\theta$



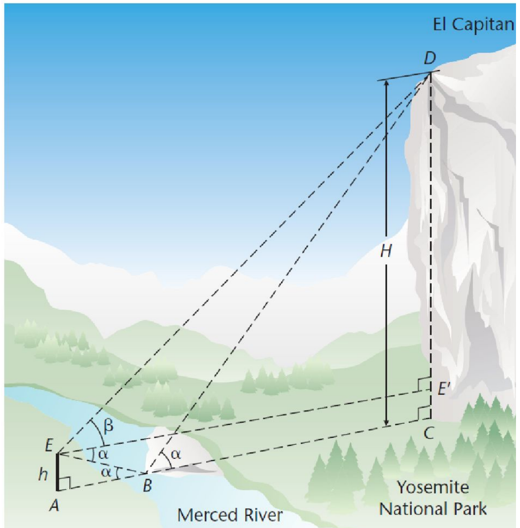
Introduction

- **Analytic trigonometry** combines the use of a coordinate system with algebraic manipulation of the various trigonometry functions to obtain formulas useful for scientific and engineering applications.

- Trigonometric identities help to solve problems in many different areas such as

- Music. Sound waves

- Sports. Path of javelin



- Surveying

Using addition formulae

$$\cos(A-B)\equiv\cos(A)\cos(B)+\sin(A)\sin(B)$$

$$\cos(A+B)\equiv\cos(A)\cos(B)-\sin(A)\sin(B)$$

$$\sin(A+B)\equiv\sin(A)\cos(B)+\cos(A)\sin(B)$$

$$\sin(A-B)\equiv\sin(A)\cos(B)-\cos(A)\sin(B)$$

$$\tan(A+B)\equiv\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(\mathrm{B})}$$

$$\tan(A-B)\equiv\frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(\mathrm{B})}$$

Cosine of difference

$$\cos(A-B) \equiv \cos(A)\cos(B)+\sin(A)\sin(B)$$

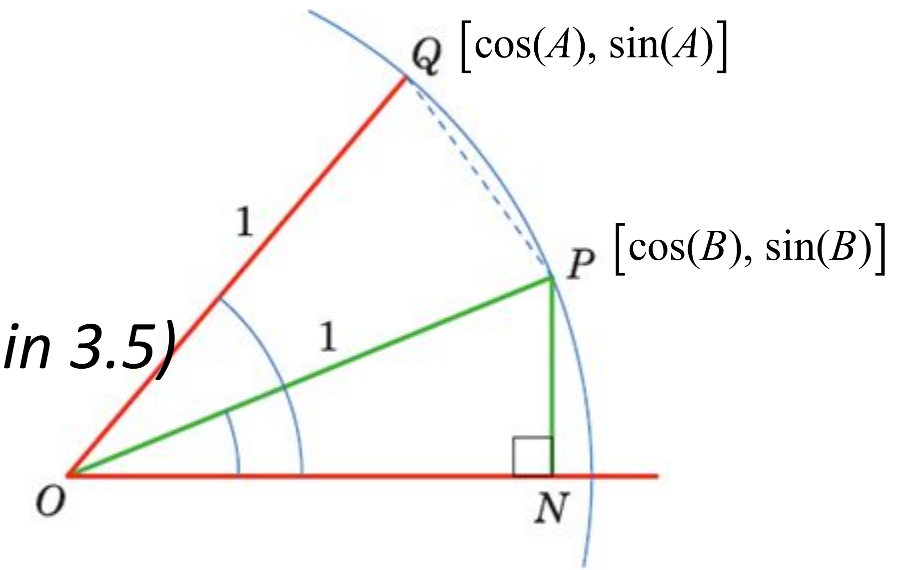
Sketch of Proof: Using the distance formula

$$\left[d(P,Q)\right]^2 = \left[\cos(A)-\cos(B)\right]^2 + \left[\sin(B)-\sin(A)\right]^2$$

$$= \cos^2(A) - 2\cos(A)\cos(B) + \cos^2(B)$$

$$+\sin^2(A) - 2\sin(A)\sin(B) + \sin^2(B)$$

$$= 2-2\left[\cos(A)\cos(B)+\sin(A)\sin(B)\right] \; *$$



Using the Law of Cosine *(the proof in 3.5)*

$$\left[d(P,Q)\right]^2 = 1^2 + 1^2 - 2\cos(A-B) \; **$$

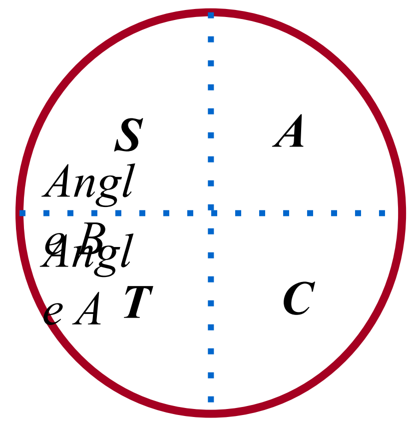
Comparing * and **, $\cos(A-B) \equiv \cos(A)\cos(B)+\sin(A)\sin(B)$

Proofs of Cosine of sum, Sine of sum and difference, tangent of sum and difference formulae are given in additional questions 1-5

Example 1: Given that sin(*A*) = – 3/5, where π < *A* < 3π/2 , and cos(*B*) = – 12/13, where *B* is obtuse, find the exact values of

$$(i)\ \cos(A-B) \qquad (ii)\ \tan(A+B)$$

$$(i)\ \cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$$



$$\cos(A)=\pm\sqrt{1-\sin^2(A)}=\pm\sqrt{1-(3/5)^2}=\pm 4/5$$

Since A is in QIII, $\cos(A)=-\dfrac{4}{5}$

$$\sin(B)=\pm\sqrt{1-\cos^2(B)}=\pm\sqrt{1-(12/13)^2}=\pm 5/13$$

Since B is in QII, $\sin(B)=\dfrac{5}{13}\Rightarrow \cos(A-B)=-\dfrac{4}{5}\left(-\dfrac{12}{13}\right)+\left(-\dfrac{3}{5}\right)\dfrac{5}{13}=\dfrac{33}{65}$

Example 1: Given that sin(*A*) = – 3/5, where π < *A* < 3π/2 , and cos(*B*) = – 12/13, where *B* is obtuse, find the exact values of

$$(ii)\ \tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$= \frac{\sin(A)/\cos(A)+\sin(B)/\cos(B)}{1-[\sin(A)/\cos(A)][\sin(B)/\cos(B)]}$$

$$= \frac{3/4-5/12}{1-3/4(-5/12)} = \frac{16}{63}$$

Your turn!

If $f(x) = \sin x$, show that

$$\frac{f(x+h)-f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

Solution

$$\frac{f(x+h)-f(x)}{h}=\frac{\sin(x+h)-\sin x}{h}$$ Definition of f

$$=\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$ Addition Formula for Sine

$$=\frac{\sin x\,(\cos h-1)+\cos x\sin h}{h}$$ Factor

$$=\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)$$ Separate the fraction

This will be used in Calculus to derive that the derivative of sine is cosine

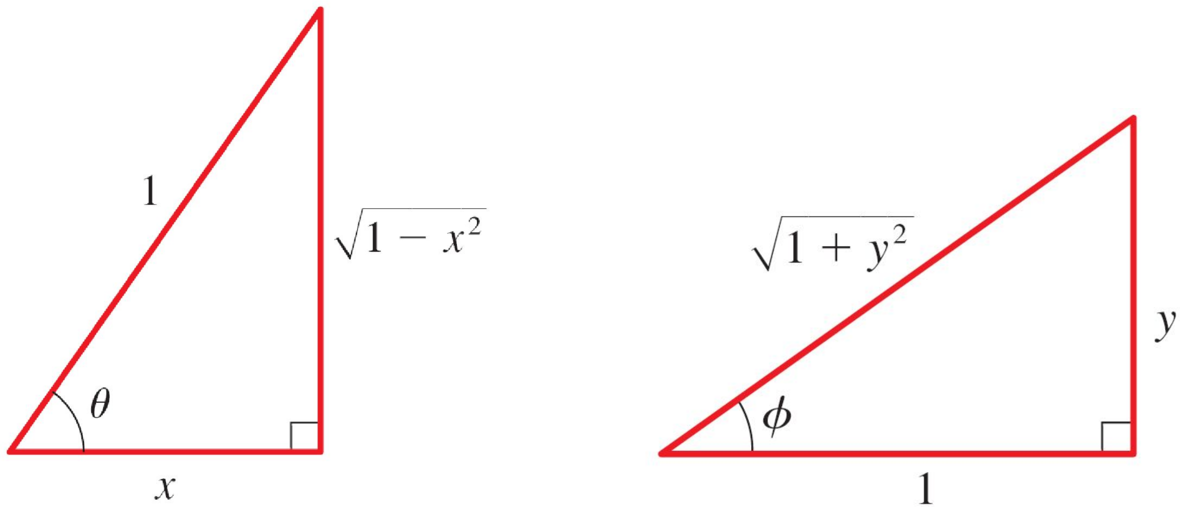
Example 2

Write $\sin(\cos^{-1} x + \tan^{-1} y)$ as an algebraic expression in x and y, where $-1 \leq x \leq 1$ and y is any real number.

Solution

Let θ = $\cos^{-1}x$ and φ = $\tan^{-1}y$.

We sketch triangles with angles θ and φ such that $\cos\theta = x$ and $\tan \varphi = y$.



$\cos\theta = x$

$\tan\varphi = y$

From the triangles we have

$\sin\theta = \sqrt{1-x^2}$ (Note, $\sin\theta > 0$ since $0 \le \theta \le \pi$)

$\cos\phi = \dfrac{1}{\sqrt{1+y^2}}$ ($\cos\phi > 0$, since $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$)

$\sin\phi = \dfrac{y}{\sqrt{1+y^2}}$ (Since $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$, y determines the sign of $\sin\phi$)

Using the Addition Formula

$$\sin(\cos^{-1}x + \tan^{-1}y) = \sin(\theta + \phi)$$

$$= \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$= \sqrt{1-x^2}\,\frac{1}{\sqrt{1+y^2}} + x\,\frac{y}{\sqrt{1+y^2}} = \frac{1}{\sqrt{1+y^2}}\left(\sqrt{1-x^2} + xy\right)$$

Double angle formulae

$$\sin(2A) \equiv 2\sin(A)\cos(A)$$

$$\cos(2A) \equiv \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)$$

$$\tan(2A) \equiv \frac{2\tan(A)}{1 - \tan^2(A)}$$

Prove double angle for sine

Prove $\sin(2A) = 2\sin(A)\cos(A)$

Using addition formula for sin(A + A):

$$\sin(2A) = \sin(A)\cos(A) + s\,\text{in}(A)\cos(A)$$

$$= 2\sin(A)\cos(A)$$

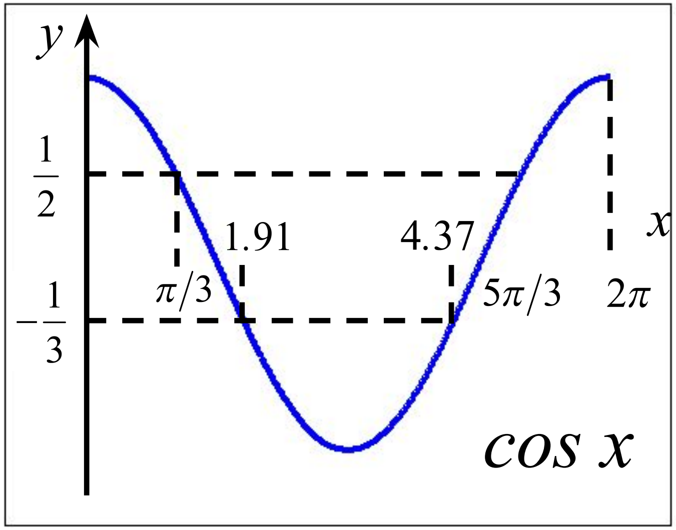
Proofs for Cosine and tangent of double angle formulae are given in additional questions 6-7

Example 3: Solve

$3\cos(2x) - \cos(x) + 2 = 0$, for $0 \le x \le 2\pi$

Using the formula for double angle

$$3\cos(2x) - \cos(x) + 2$$
$$= 3\left[2\cos^2(x) - 1\right] - \cos(x) + 2 = 0$$
$$\Rightarrow 6\cos^2(x) - \cos(x) - 1 = 0$$
$$\Rightarrow \left[3\cos(x) + 1\right]\left[2\cos(x) - 1\right] = 0$$



1. $\cos(x) = -\frac{1}{3} \to x = \arccos(-\frac{1}{3})$, $0 \le x \le 2\pi \to$ $x \approx 1.91, x \approx 4.37$

2. $\cos(x) = \frac{1}{2} \to x = \arccos(\frac{1}{3})$, $0 \le x \le 2\pi \to$ $x = \frac{\pi}{3}, x = \frac{5\pi}{3}$

Example 4: By writing the following equations as quadratic in tan(*x/2*) solve

$\sin(x)+2\cos(x)=1$, for $0 \le x \le 2\pi$

Using the double angle formulae and Pythagorean identity:

$$\sin(x)+2\cos(x)=1 \Leftrightarrow \sin(2\frac{x}{2})+2\cos(2\frac{x}{2})=1$$

$$\Leftrightarrow 2\cos(x/2)\sin(x/2)+2\cos^2(x/2)-2\sin^2(x/2)=\sin^2(x/2)+\cos^2(x/2)$$

$$\Leftrightarrow 2\cos(x/2)\sin(x/2)+\cos^2(x/2)-3\sin^2(x/2)=0$$

Dividing by $3\cos^2(x/2)$ and rearranging

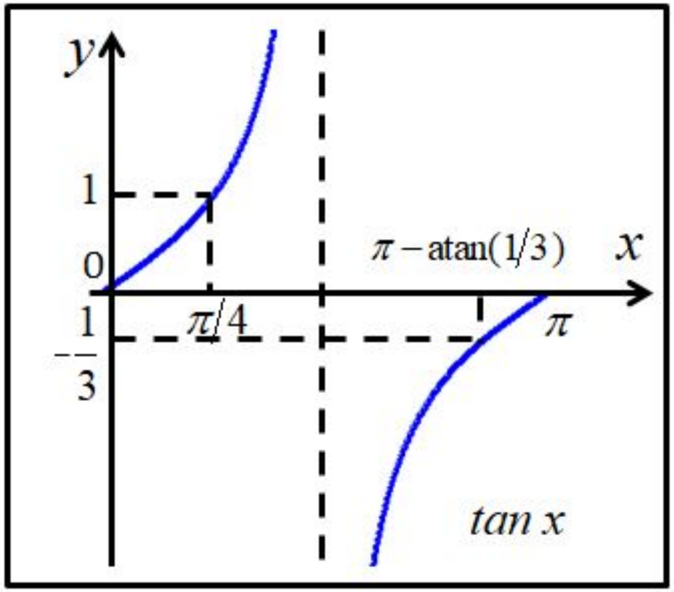
$$\tan^2(x/2)-(2/3)\tan(x/2)-(1/3)=0$$

$$\Leftrightarrow \tan(x/2)=(1/2)\left[(2/3)\pm\sqrt{(4/9)+(12/9)}\right]$$

$$\Leftrightarrow \tan(x/2)=-1/3 \text{ or } \tan(x/2)=1$$

$$\Leftrightarrow x/2=\pi/4 \text{ or } x/2=\pi+\arctan(-1/3)$$

$$\Leftrightarrow x=\pi/2 \text{ or } x\approx 5.64$$



Note: If $0\le x\le 2\pi$,
then $0\le \frac{x}{2}\le \pi$

Your turn!

Solve $\sin(2x) = \sin(x)$ for $0 \le x \le 2\pi$.

Your turn!

Solve $\sin(2x) = \sin(x)$ for $0 \le x \le 2\pi$

Using $\sin(2x)$ formula:

$$\sin(2x) = 2\sin(x)\cos(x)$$

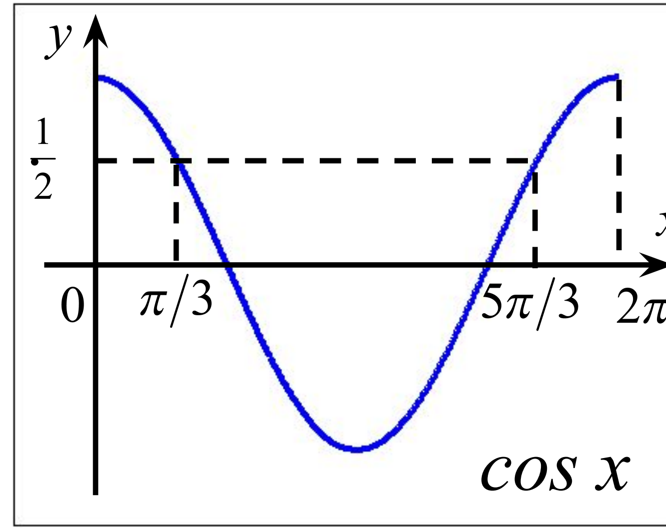
$$\Leftrightarrow 2\cos(x)\sin(x) = \sin(x)$$

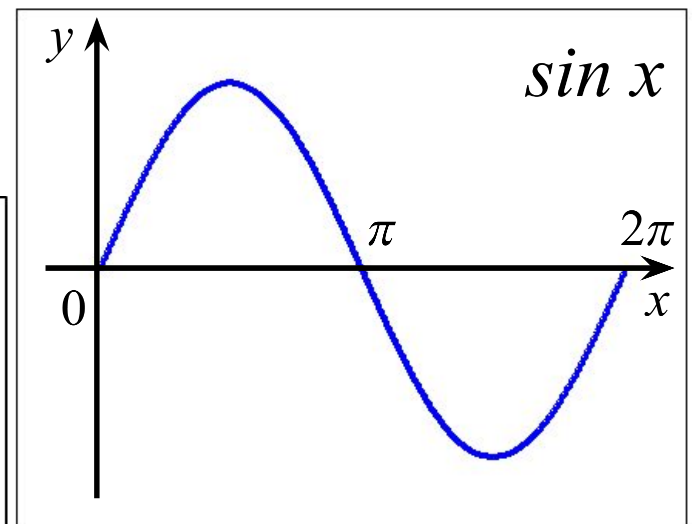
$$\Leftrightarrow \sin(x)\left[2\cos(x) - 1\right] = 0$$

$$\Leftrightarrow \sin(x) = 0 \text{ or } \cos(x) = 1/2$$

1. $\sin(x) = 0$
$\rightarrow x = 0, \pi, 2\pi.$
$0 \le x \le 2\pi$

2. $\cos(x) = \frac{1}{2} \rightarrow x =$
$\arccos(\frac{1}{3}), 0 \le x \le$
$2\pi \rightarrow x = \frac{\pi}{3}, x = \frac{5\pi}{3}$





Using double angle formulae to prove identities

Example 5: Prove $\tan(2x) \equiv \dfrac{2}{\cot(x) - \tan(x)}$

$$
\begin{aligned}
\tan(2x) &= \frac{2\tan(x)}{1 - \tan^2(x)} \\
&= \frac{2}{1/\tan(x) - \tan(x)} \\
&= \frac{2}{\cot(x) - \tan(x)}
\end{aligned}
$$

Example 6: Prove $\sin(x) \equiv \dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$

$$\frac{2\tan(x/2)}{1+\tan^2(x/2)} = \frac{2\dfrac{\sin(x/2)}{\cos(x/2)}}{1+\dfrac{\sin^2(x/2)}{\cos^2(x/2)}}$$

$$= \frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)+\sin^2(x/2)}$$

$$= \sin(x)$$

Your turn!

Prove $\sin(3A) \equiv 3\sin(A) - 4\sin^3(A)$

Your turn!

Prove $\sin(3A) \equiv 3\sin(A) - 4\sin^3(A)$

$$\begin{aligned}
\sin(3A) &= \sin(2A + A) \\
&= \sin(2A)\cos(A) + \cos(2A)\sin(A) \\
&= 2\sin(A)\cos(A)\cos(A) + \left[\cos^2(A) - \sin^2(A)\right]\sin(A) \\
&= 3\sin(A)\cos^2(A) - \sin^3(A) \\
&= 3\sin(A)\left[1 - \sin^2(A)\right] - \sin^3(A) \\
&= 3\sin(A) - 4\sin^3(A)
\end{aligned}$$

Half-angle formulas

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Proof: Using double angle formula for cosine

$\cos 2x = 1 - 2\sin^2 x \rightarrow$

$$\rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

Homework: prove for cosine and tangent

Half-angle formulas

Using formulas for lowering powers

HALF-ANGLE FORMULAS

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

Example 7

Find the exact value of sin 22.5°.

Solution:

Since 22.5° is half of 45°, we use the Half-Angle Formula for Sine with *u* = 45°. We choose the + sign because 22.5° is in the first quadrant:

$$\sin\frac{45^\circ}{2} = \sqrt{\frac{1-\cos 45^\circ}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1-\sqrt{2}/2}{2}} \qquad \cos 45^\circ = \sqrt{2}/2$$

$$= \tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

Using R-formulae to solve equations

Expressions like

$$a\cos x + b\sin x\ ,\quad a, b > 0$$

can be written as a single sine or cosine of the form

$$R\sin(x+\alpha)\ \text{ or } R\cos(x-\alpha)$$

where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$

A key idea is to use the sum (or difference) formula of sine or cosine.

We will study how to find R and α by examples.

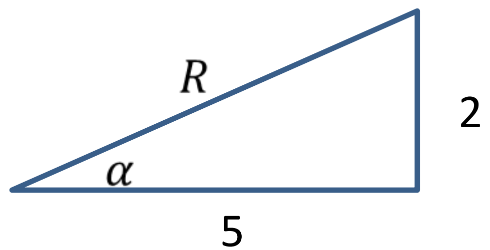
Example 8:

Express $2\cos x + 5\sin x$ as $R\sin(x+\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$

Solution

$$2\cos x + 5\sin x = R\sin x\cos\alpha + R\cos x\sin\alpha$$

Comparing the two sides, we have $\cos\alpha = \frac{5}{R}$ and $\sin\alpha = \frac{2}{R}$



Hence, $R = \sqrt{2^2+5^2} = \sqrt{29}$ and $\alpha = \arctan\frac{2}{5} = 0.38$ (2 dp)

$$2\cos x + 5\sin x = \sqrt{29}\sin(x+0.38)$$

In the previous example, we can also use the difference formula of cosine, $R\cos(x-\alpha)$.

However, we cannot use $R\sin(x-\alpha)$ or $R\cos(x+\alpha)$ because of the "+" sign between $2\cos x$ and $5\sin x$.

You need to look at the sign between the two terms and determine which form you should use.

Generally speaking, for $a, b > 0$

$$a\sin x \pm b\cos x = R\sin\left(x \pm \arctan\frac{b}{a}\right), \ R = \sqrt{a^2+b^2}$$

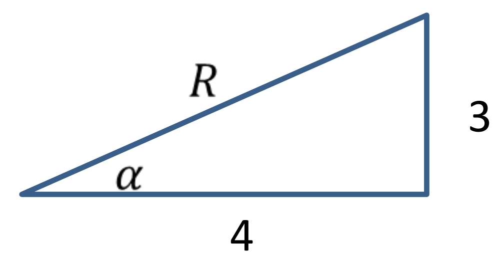
Example 9: Solve $4\cot(x) - 3 = -5\operatorname{cosec}(x)$ for $-\pi < x < \pi$

$$4\cot(x) - 3 = -5\operatorname{cosec}(x) \Rightarrow 4\cos(x) - 3\sin(x) = -5$$

Express $4\cos x - 3\sin x$ as a single sine or cosine. Here, let's use cosine. (You can also use sine, see slide 31)

$$4\cos x - 3\sin x = R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$$

$$\cos\alpha = \frac{4}{R}, \qquad \sin\alpha = \frac{3}{R}$$



$$R = \sqrt{3^2+4^2} = 5, \qquad \alpha = \arctan\frac{3}{4}$$

Now solve the equation

$$5\cos\left(x + \arctan\frac{3}{4}\right) = -5$$

$$x + \arctan\frac{3}{4} = \pi + 2n\pi\,, \qquad n \in Z$$

$$x = -\arctan\frac{3}{4} + \pi + 2n\pi$$

$x = 2.5$ (1 dp)

Alternative expression for the previous example

$$4\cos x - 3\sin x = -(3\sin x - 4\cos x) = -R\sin(x-\alpha)$$

$$= -(R\sin x\cos\alpha - R\cos x\sin\alpha)$$

$$\cos\alpha = \frac{3}{R}, \sin\alpha = \frac{4}{R}$$

$R = 5,\ \alpha = \arctan\frac{4}{3} = 0.93$ (2 dp)

$$4\cos x - 3\sin x = -5\sin(x - 0.93)$$

Example 10

Without differentiating find the maximum value of

$$12\cos(x)+5\sin(x)$$

and smallest positive value of x at which it arises.

Example 10. Solution

Without differentiating find the maximum value of

$$12\cos(x)+5\sin(x)$$

and smallest positive value of x at which it arises.

$$12\cos(x)+5\sin(x)=\sqrt{12^2+5^2}\cos\left[x-\arctan\left(5/12\right)\right]$$

$$\approx 13\cos\left(x-0.39\right)$$

Therefore, the maximum value is 13. It is attained when

$$\cos(x-0.39)=1$$

that is, when $x\approx 0.39$

Your turn!

Solve $\cos(2x)+\sin(2x)=1$ for $0 \le x < 2\pi$

Your turn!

Solve $\cos(2x)+\sin(2x)=1$ for $0\le x<2\pi$

$$\cos(2x)+\sin(2x)=\sqrt{1^2+1^2}\,\sin\left[2x+\arctan(1)\right]=\sqrt{2}\,\sin(2x+\pi/4)$$

$$\sqrt{2}\sin(2x+\pi/4)=1 \Rightarrow \sin(2x+\pi/4)=1/\sqrt{2}$$

$$\Rightarrow 2x+\pi/4=\arcsin\left(1/\sqrt{2}\right)+2\pi n=\pi/4+2\pi n\,,\quad n\in\mathbb{Z}$$

$$\Rightarrow x=\pi n\,,\quad n\in\mathbb{Z}$$

OR

$$\Rightarrow 2x+\pi/4=\pi-\arcsin\left(1/\sqrt{2}\right)+2\pi n=3\pi/4+2\pi n,\quad n\in\mathbb{Z}$$

$$\Rightarrow x=\pi/4+\pi n,\quad n\in\mathbb{Z}$$

But since $0\le x<2\pi$, the only solutions are

$x\in\{0,\pi/4,\pi,5\pi/4\}$

Using factor formulae

$$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u+v) + \cos(u-v)]$$

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$

Prove

$$\sin u \ \cos v \ = \frac{1}{2}[\sin(u+v)+\sin(u \ - v)]$$

- Using addition formulae

(1) $\sin(u \ + \ v) \ = \ \sin u \cos v \ + \ \cos u \sin v$

$(1) + (2)$

(2) $\sin(u - \ v) \ = \ \sin u \cos v - \ \cos u \sin v$

$\sin(u \ + \ v) + \ \sin(u - \ v) \ = \ 2 \sin u \cos v \mid 2$

$$\sin u \ \cos v \ = \frac{1}{2}[\sin(u+v)+\sin(u \ - v)]$$

Sum-to-Product formulas

Sums and differences of sines and cosines can be expressed as products of sines and/or cosines by using the “factor formulae”:

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

Prove

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

- Using the first product-to-sum formula

$$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$

Let $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$. Substitute,

$$\sin\frac{x+y}{2}\cos\frac{x-y}{2} = \frac{1}{2}(\sin x + \sin y)$$

$$\Rightarrow \sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

Example 11

- Find the value of the product

$$3\cos 37.5^\circ \cos 7.5^\circ$$

Example 11. Solution

- $3\cos 37.5° \cos 7.5° = 3[\frac{1}{2}(\cos 45° + \cos 30°)]$

$$= \frac{3}{2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right] = \frac{3}{4}(\sqrt{2} + \sqrt{3})$$

Example 12:

Solve $\sin(4x)-\sin(3x)=0$ for $0\le x\le\pi$

$$\sin(4x)-\sin(3x)=2\cos\left(7x/2\right)\sin\left(x/2\right)=0$$

$$\left.\begin{array}{l}\Rightarrow\cos\left(7x/2\right)=0\\ 0\le x\le\pi\ \Rightarrow 0\le 7x/2\le 7\pi/2\end{array}\right\}\Rightarrow\frac{7x}{2}\in\left\{\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}\right\}$$

$$\Rightarrow x\in\left\{\frac{\pi}{7},\frac{3\pi}{7},\frac{5\pi}{7},\pi\right\}$$

OR

$$\left.\begin{array}{l}\Rightarrow \sin\left(x/2\right)=0 \\ 0\le x\le\pi \Rightarrow 0\le x/2\le \pi/2\end{array}\right\}\Rightarrow \frac{x}{2}=0$$

$$\Rightarrow x=0$$

$$\Rightarrow x \in \{0\}\cup\left\{\frac{\pi}{7},\frac{3\pi}{7},\frac{5\pi}{7},\pi\right\}=\left\{0,\frac{\pi}{7},\frac{3\pi}{7},\frac{5\pi}{7},\pi\right\}$$

Example 13

Prove that $$\frac{\sin(x+2y)+\sin(x+y)+\sin(x)}{\cos(x+2y)+\cos(x+y)+\cos(x)}=\tan(x+y)$$

Using factor formula for sines

$$\sin(x+2y)+\sin(x)=2\sin(x+y)\cos(y)$$

Using factor formula for cosines

$$\cos(x+2y)+\cos(x)=2\cos(x+y)\cos(y)$$

$$\Rightarrow \frac{\sin(x+2y)+\sin(x+y)+\sin(x)}{\cos(x+2y)+\cos(x+y)+\cos(x)}=\frac{\sin(x+y)\left[1+2\cos(y)\right]}{\cos(x+y)\left[1+2\cos(y)\right]}=\tan(x+y)$$

Your turn!

Prove that $$\frac{\sin(9x)-\sin(x)}{\cos(12x)+\cos(2x)}=\frac{\sin(4x)}{\cos(7x)}$$

Your turn!

Prove that $\dfrac{\sin(9x)-\sin(x)}{\cos(12x)+\cos(2x)} = \dfrac{\sin(4x)}{\cos(7x)}$

Using factor formula for sines

$$\sin(9x)-\sin(x) = 2\sin(4x)\cos(5x)$$

Using factor formula for cosines

$$\cos(12x)+\cos(2x) = 2\cos(7x)\cos(5x)$$

$$\Rightarrow \frac{\sin(9x)-\sin(x)}{\cos(12x)+\cos(2x)} = \frac{2\sin(4x)\cos(5x)}{2\cos(7x)\cos(5x)} = \frac{\sin(4x)}{\cos(7x)}$$

Additional Question 1

Prove $\cos(A+B) \equiv \cos(A)\cos(B) - \sin(A)\sin(B)$

$$\begin{aligned}\cos(A+B) &= \cos\left[A-\left(-B\right)\right]\\ &= \cos(A)\cos(-B) + \sin(A)\sin(-B)\\ &= \cos(A)\cos(B) - \sin(A)\sin(B)\end{aligned}$$

Additional Question 2

Prove $\sin(A+B) \equiv \sin(A)\cos(B) + \cos(A)\sin(B)$

$$\begin{aligned}
\sin(A+B) &= \cos\left[\frac{\pi}{2} - (A+B)\right] \\
&= \cos\left[\left(\frac{\pi}{2} - A\right) - B\right] \\
&= \cos\left(\frac{\pi}{2} - A\right)\cos(B) + \sin\left(\frac{\pi}{2} - A\right)\sin(B) \\
&= \sin(A)\cos(B) + \cos(A)\sin(B)
\end{aligned}$$

Additional Question 3

Prove $\sin\left(A-B\right) \equiv \sin(A)\cos(B)-\cos(A)\sin(B)$

$$\sin\left(A-B\right) = \sin\left[A+\left(-B\right)\right]$$

$$= \sin(A)\cos(-B)+\cos(A)\sin(-B)$$

$$= \sin(A)\cos(B)-\cos(A)\sin(B)$$

Additional Question 4

Prove $\tan\left(A+B\right) \equiv \dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$

$$\tan\left(A+B\right) = \frac{\sin\left(A+B\right)}{\cos\left(A+B\right)}$$

$$= \frac{\sin(A)\cos(B)+\cos(A)\sin(B)}{\cos(A)\cos(B)-\sin(A)\sin(B)}$$

$$= \frac{\dfrac{\sin(A)\cos(B)}{\cos(A)\cos(B)}+\dfrac{\cos(A)\sin(B)}{\cos(A)\cos(B)}}{\dfrac{\cos(A)\cos(B)}{\cos(A)\cos(B)}-\dfrac{\sin(A)\sin(B)}{\cos(A)\cos(B)}} = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

Additional Question 5

Prove $\tan\left(A-B\right) \equiv \dfrac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$

$$\tan\left(A-B\right) = \tan\left[A+\left(-B\right)\right]$$

$$= \frac{\tan(A)+\tan(-B)}{1-\tan(A)\tan(-B)}$$

$$= \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

Additional Question 6

Prove $\cos(2A) = \cos^2(A) - \sin^2(A)$

$$\cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1$$

$$\cos^2(A) - \sin^2(A) = 1 - 2\sin^2(A)$$

Additional Question 6

$$\cos(2A) = \cos(A)\cos(A) - \sin(A)\sin(A)$$
$$= \cos^2(A) - \sin^2(A)$$

$$\cos^2(A) - \sin^2(A) = \cos^2(A) - \left[1 - \cos^2(A)\right] = 2\cos^2(A) - 1$$

$$\cos^2(A) - \sin^2(A) = \left[1 - \sin^2(A)\right] - \sin^2(A) = 1 - 2\sin^2(A)$$

Additional Question 7

Prove $\tan(2A) = \dfrac{2\tan(A)}{1-\tan^2(A)}$

$$\tan(2A) = \frac{\tan(A)+\tan(A)}{1-\tan(A)\tan(A)}$$

$$= \frac{2\tan(A)}{1-\tan^2(A)}$$

Additional Question 8(a)

Prove that expressions of the form

$$a\,cos\,(x) \pm b\sin(x)$$

where a>0 and b>0, can be expressed in the following form

$$R\cos(x \square \alpha), \quad \text{where} \quad R = \sqrt{a^2 + b^2}, \quad \text{and} \quad \alpha = \arctan(b/a)$$

Additional Question 8(a)

$$a\cos(x) \pm b\sin(x) =$$

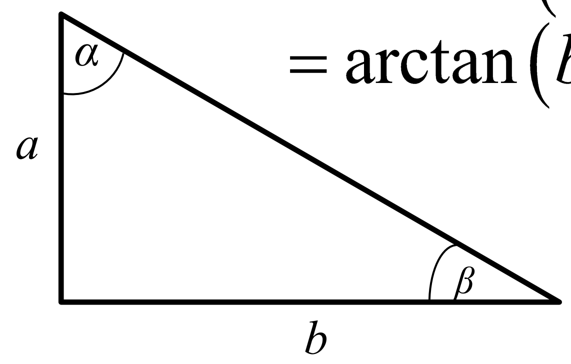
$$\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(x) \pm \frac{b}{\sqrt{a^2+b^2}}\sin(x)\right] =$$

$$\sqrt{a^2+b^2}\left[\cos\left(\alpha\right)\cos(x) \pm \sin\left(\alpha\right)\sin(x)\right]$$

Using difference formula for cosine

$$= \sqrt{a^2+b^2}\cos\left[x \mp \arctan\left({}^{b}/_{a}\right)\right]$$

$$\alpha = \operatorname{arccot}\left(a/b\right)$$
$$= \arctan\left(b/a\right)$$



Additional Question 8(b)

Prove that expressions of the form

$$a\,sin\,(x) \pm b\cos(x)$$

where a>0 and b>0, can be expressed in the following form

$$R\sin(x \pm \beta), \quad \text{where} \quad R = \sqrt{a^2 + b^2}, \quad \text{and} \quad \beta = \arctan(b/a)$$

Additional Question 8(b)

$$a\sin(x) \pm b\cos(x) =$$

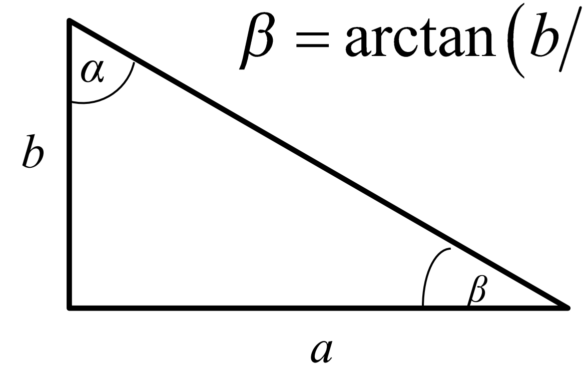
$$\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin(x) \pm \frac{b}{\sqrt{a^2+b^2}}\cos(x)\right] =$$

$$\sqrt{a^2+b^2}\left[\cos(\beta)\sin(x) \pm \sin(\beta)\cos(x)\right]$$

Using addition formula for sine

$$= \sqrt{a^2+b^2}\,\sin\left[x \pm \arctan\left(b/a\right)\right]$$

$$\beta = \arctan\left(b/a\right)$$



Learning outcomes

3.4.1 Apply the addition and subtraction formulas, the half angle and double angle formulas, R-formula, the product to sum formulas and the sum to product formulas to simplify expressions or prove identities

3.4.2 Solve trigonometric equations involving multiple angles

Preview activity: 3.5 Trigonometry 5

Hearing Test

People can hear sounds at frequencies from about 20 Hz to 20,000 Hz, though we hear sounds best from 1,000 Hz to 5,000 Hz. Dogs can hear from about 64 Hz to 44,000 Hz!

In this video, you can hear the sound at frequencies from 20 Hz to 20,000 Hz and see their corresponding waves.

Do the quick test and see how much you can hear.

Also, observe how the sound and its wave change when the frequency becomes higher.