# Graphs

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- The shortest-path problem
- Dijkstra’s algorithm

# Weighted graphs

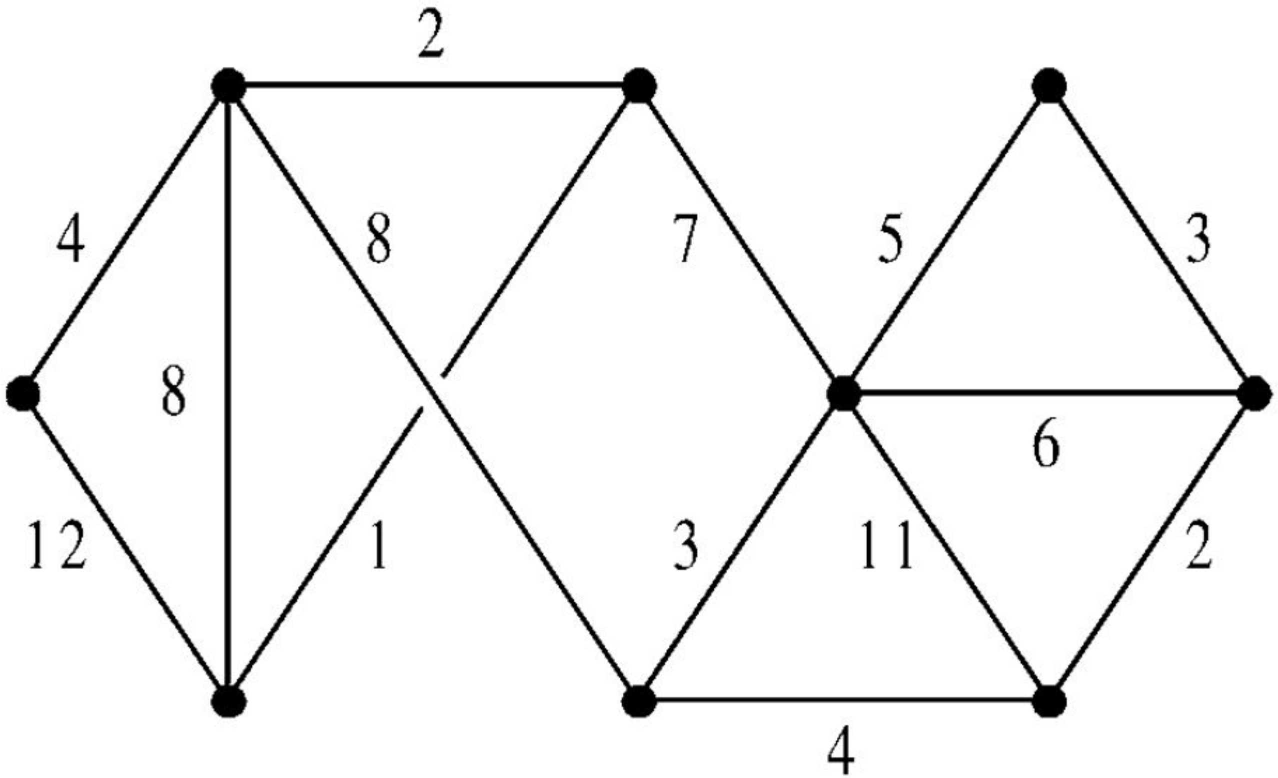
**Definition 5**

A **weighted graph** is a graph $\Gamma$ together with a function $w\colon E_\Gamma \to \mathbb{R}^+ \cup \{0\}$.

If $e$ is an edge of $\Gamma$, then the number $w(e)$ is called the **weight** of $e$.

## Weighted graphs

For example, the figure is a diagram of a weighted graph.



# Weighted graphs

**Definition 1**

Let $\Gamma$ be a weighted graph.

For any subgraph $\Gamma'$ of $\Gamma$ we define the **weight** of $\Gamma'$, $w(\Gamma')$, to be the sum of the weights of its edges.

Symbolically,

$$w(\Gamma') = \sum_{e \in E_{\Gamma'}} w(e) .$$

# The shortest-path problem

- Let $\Gamma$ be a connected weighted graph and let $v$, $v'$ be two vertices of $\Gamma$. The **shortest-path problem** is to find a path joining $v$ and $v'$ with the smallest weight.

Of course, since a path is a subgraph of $\Gamma$, its weight is defined as the sum of the weights of its edges.

As we are dealing with finite graphs, it should be obvious that a shortest path exists, although there may be more than one shortest path joining a given pair of vertices.

# The shortest-path problem

There are various methods for finding a shortest path between two given vertices.

We describe an algorithm which, like the minimal spanning tree algorithm of the previous section, constructs the path one edge at a time.

# The shortest-path problem

- The **idea** is to begin at the vertex $v$ and move through the graph assigning a number $L(u)$ to each vertex $u$ in turn which represents the length of the shortest path yet discovered from $v$ to $u$.

These 'length numbers' $L(u)$ are initially considered temporary and may subsequently be changed if we discover a path from $v$ to $u$ which has length less than the currently assigned value $L(u)$.

# The shortest-path problem

- The algorithm is detailed more precisely below.

It actually constructs a subtree of the graph containing the vertices $v$ and $v'$; a shortest path between the two vertices is then the unique path in this tree joining them.

Note that the subtree constructed by this algorithm need not be a spanning subtree of the graph – the algorithm stops as soon as a shortest path joining $v$ and $v'$ has been found.)

# Dijkstra's algorithm for finding a shortest path between two given vertices

1. First assign $L(v) = 0$ to the starting vertex $v$. We say that the vertex $v$ has been labelled with the value $0$. Furthermore, this label is permanent as we will not subsequently change its value. Since we are constructing a sequence of trees, we also begin with the tree consisting of vertex $v$ only and no edges.

# Dijkstra's algorithm for finding a shortest path between two given vertices

2. Let $u$ be the vertex which has most recently been given a permanent label. (Initially $v = u$ as this is the only vertex with a permanent label.) Consider each vertex $u'$ adjacent to $u$ in turn, and give it a temporary label as follows. (Only those vertices $u'$ without a permanent label are considered.)
   a) If $u'$ is unlabelled, then set $L(u')$ equal to $L(u) + w(e)$ where $e$ is the edge joining $u$ to $u'$. (If there is more than one such edge $e$, choose the one with the smallest weight.)

# Dijkstra's algorithm for finding a shortest path between two given vertices

- b) If $u'$ is already labelled, then again calculate $L(u) + w(e)$ as above and if this is less than the current value of $L(u')$ then change $L(u')$ to $L(u) + w(e)$; otherwise leave $L(u')$ unchanged.

# Dijkstra's algorithm for finding a shortest path between two given vertices

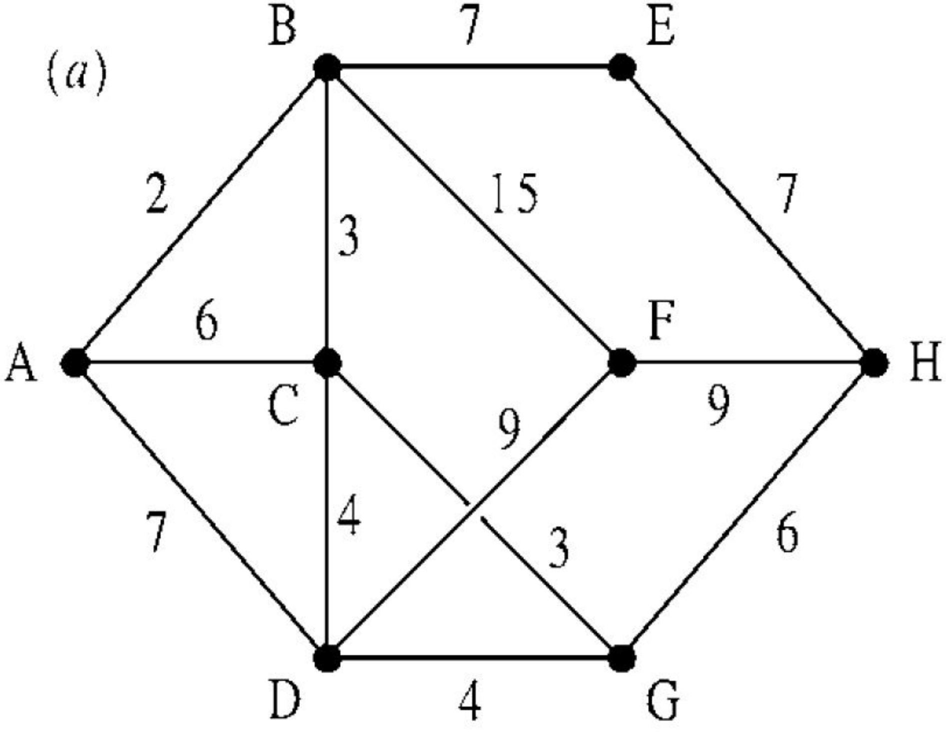
3. Choose a vertex $w$ with the smallest temporary label and make the label permanent. (There may be a choice to be made here as several temporarily labelled vertices may have equal smallest labels. It is also important to realize that $w$ need not be adjacent to $u$, the most recently labelled vertex.) At the same time adjoin to the tree so far formed the edge which gives rise to the value $L(w)$.

# Dijkstra's algorithm for finding a shortest path between two given vertices

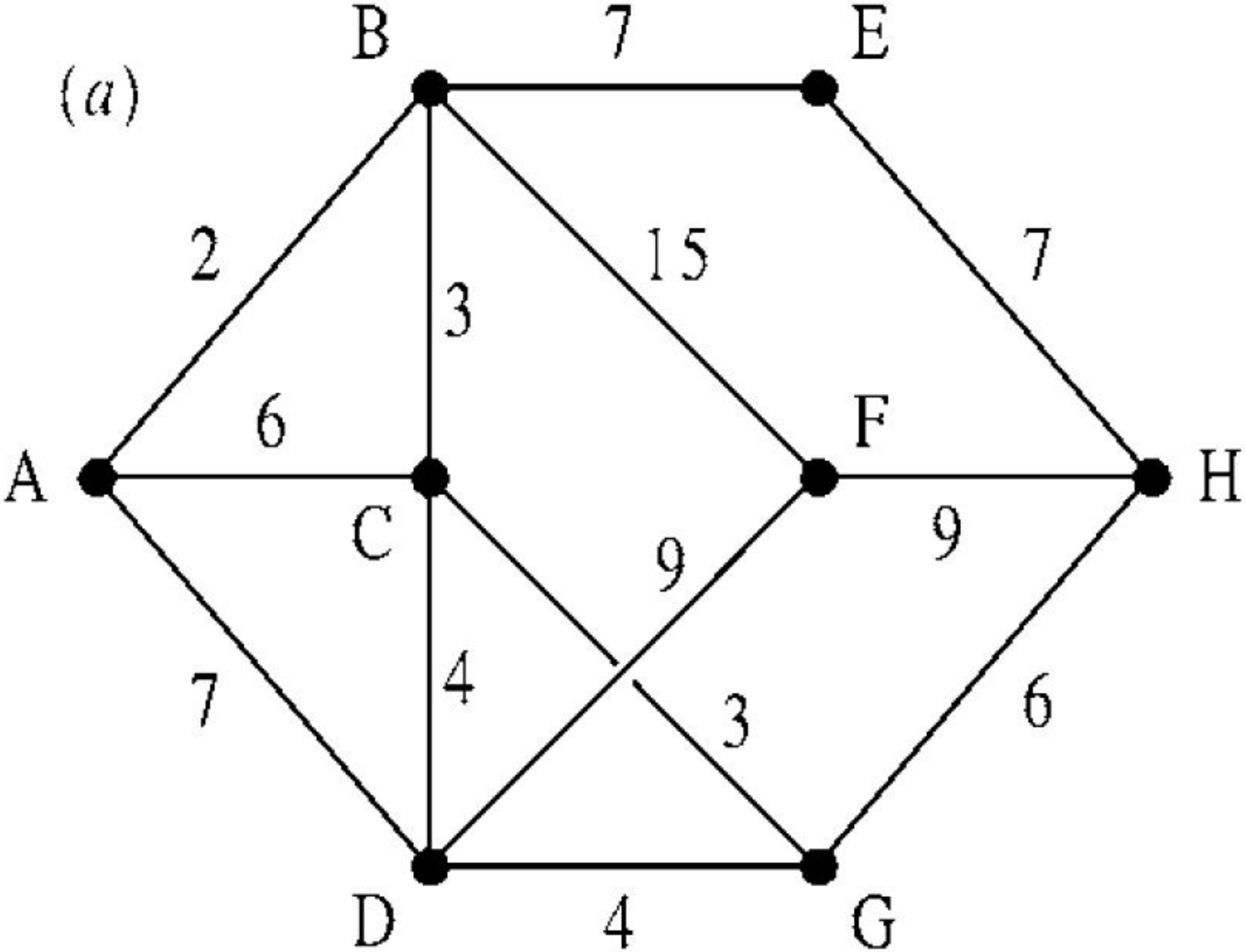
4. Repeat steps 2 and 3 until the final vertex $v'$ has been given a permanent label. The path of shortest length from $v$ to $v'$ is then the unique path in the tree thus formed joining $v$ and $v'$. Its length is the permanent value of $L(v')$.

# Dijkstra's algorithm for finding a shortest path between two given vertices

- We illustrate this algorithm by constructing a shortest path from $A$ to $H$ in the weighted graph illustrated in the figure.



# Dijkstra's algorithm



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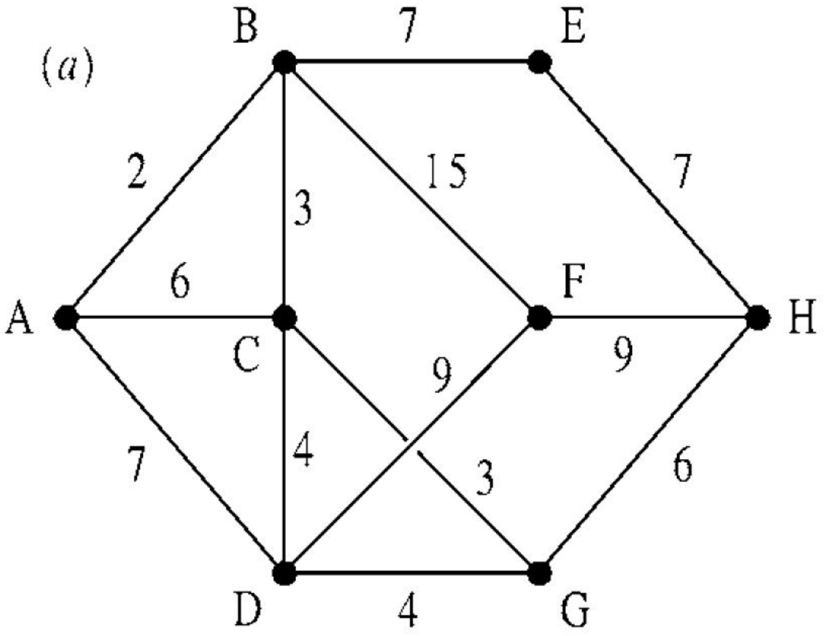


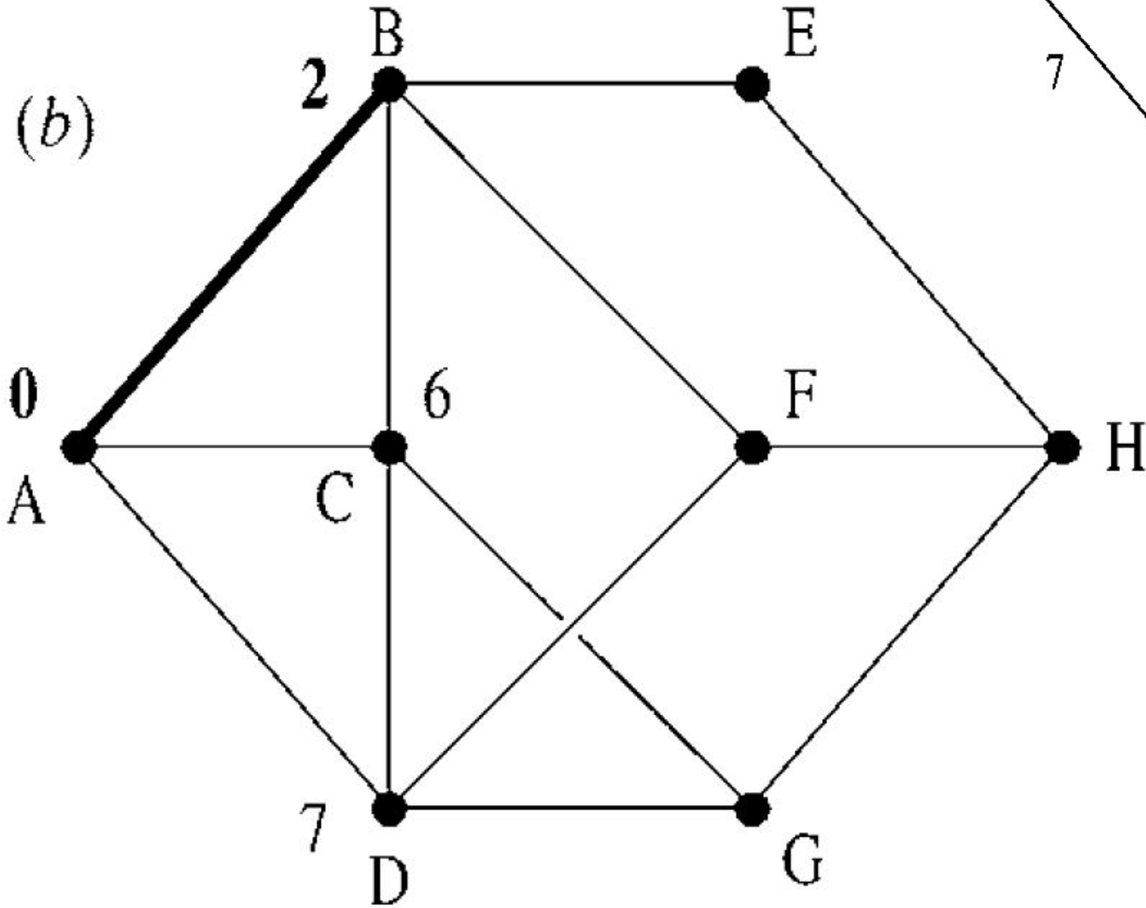

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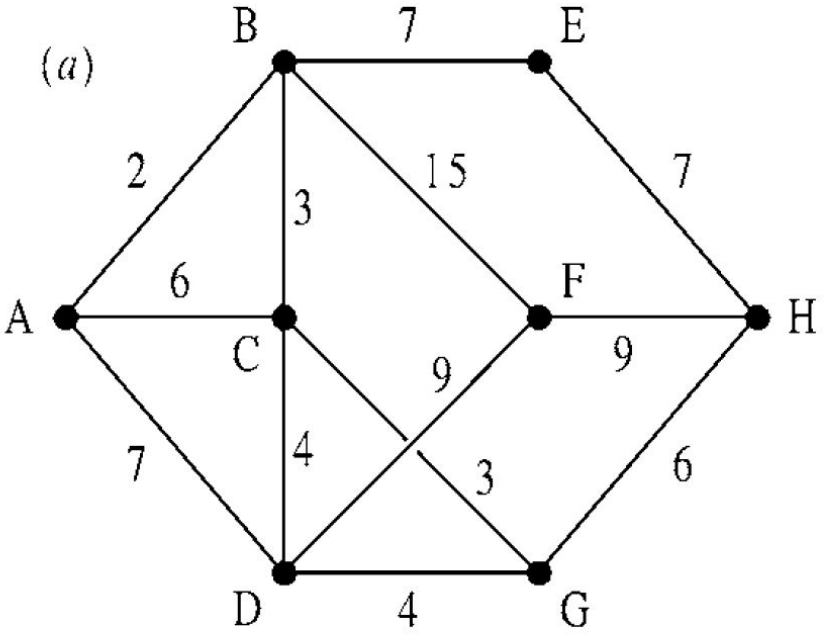


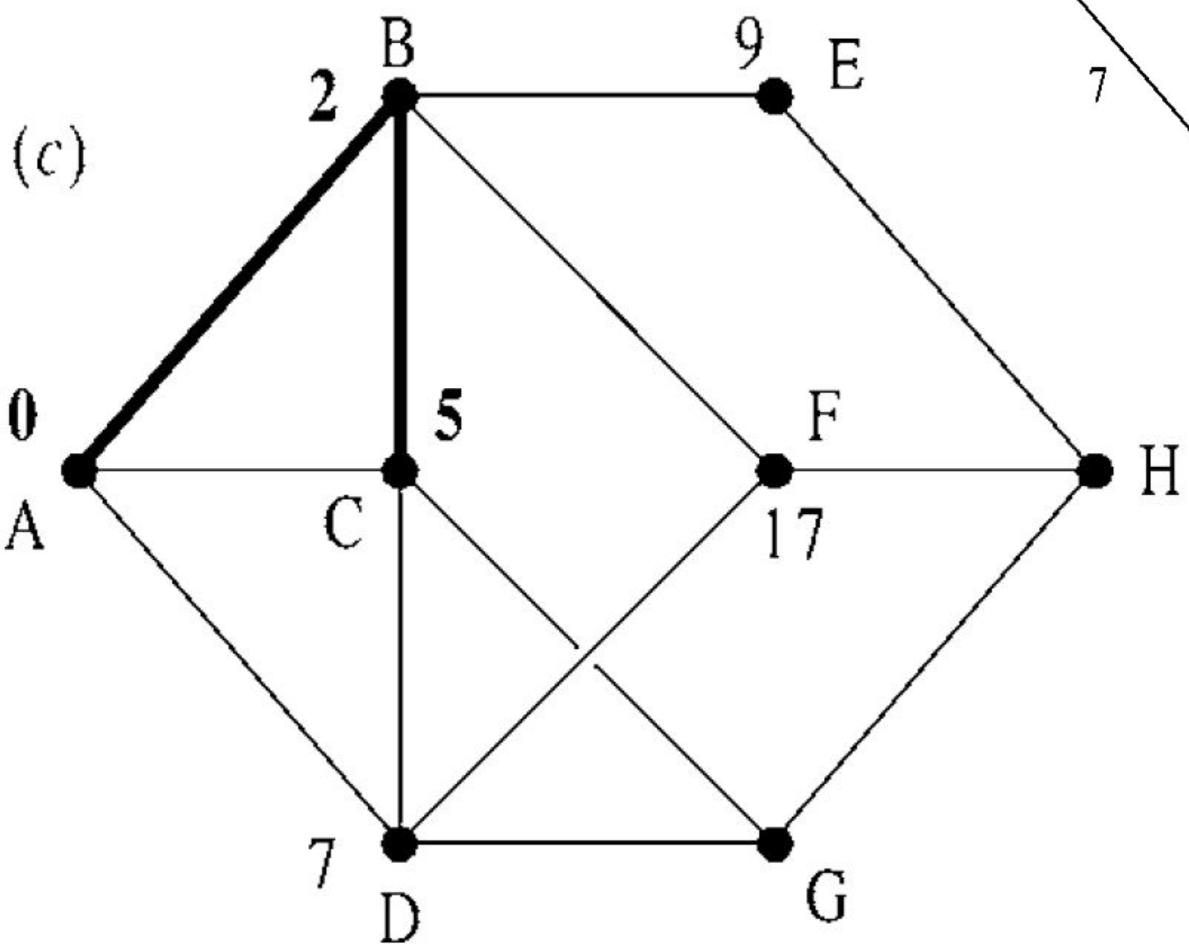

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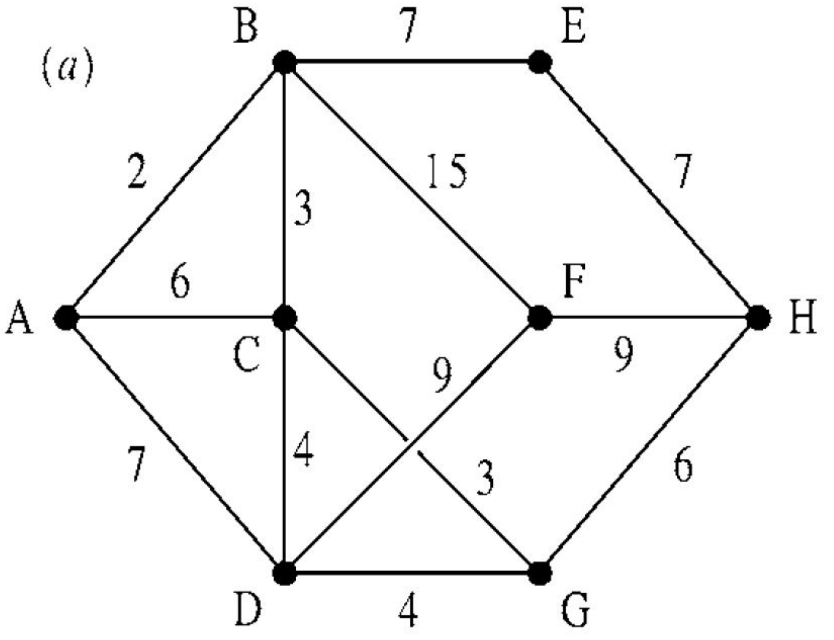


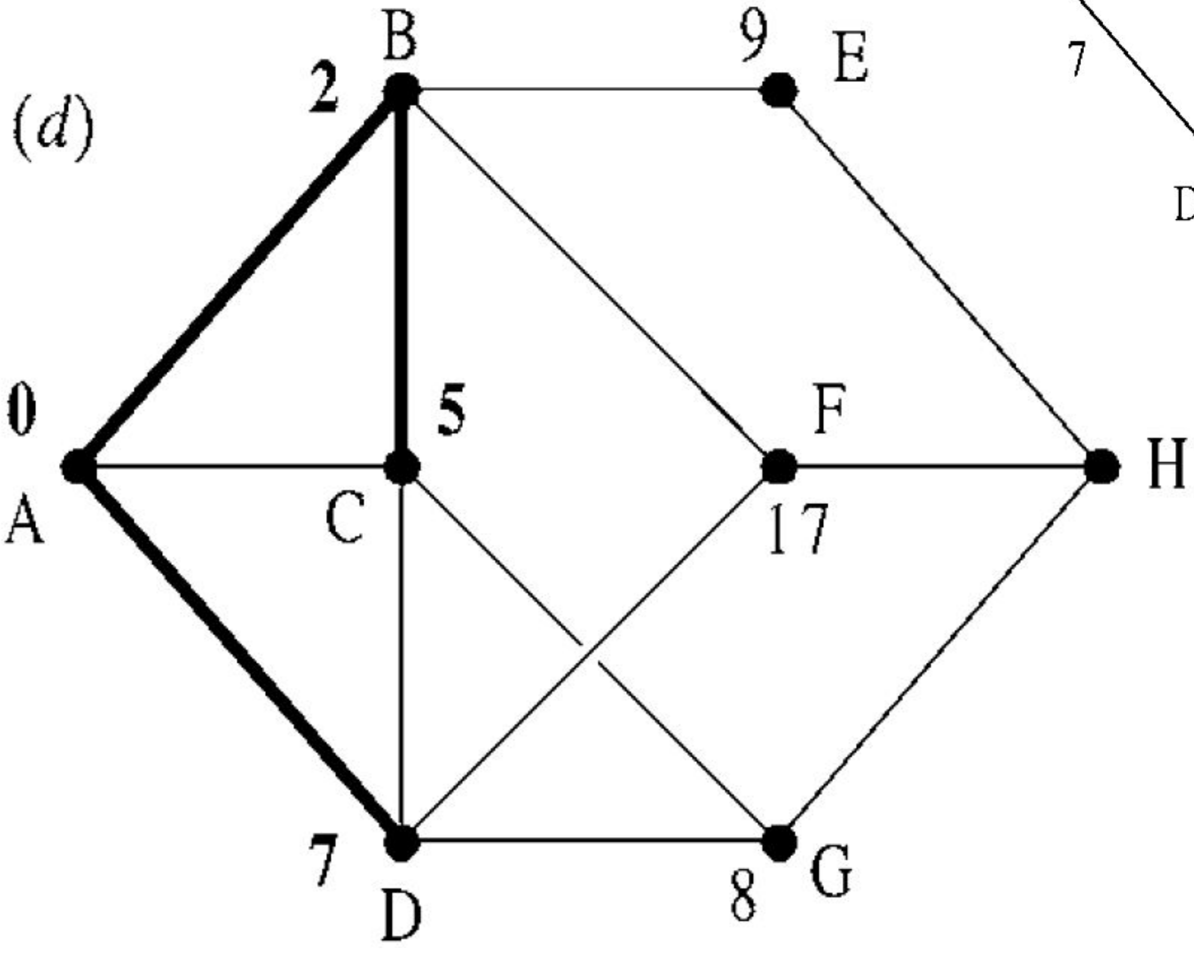

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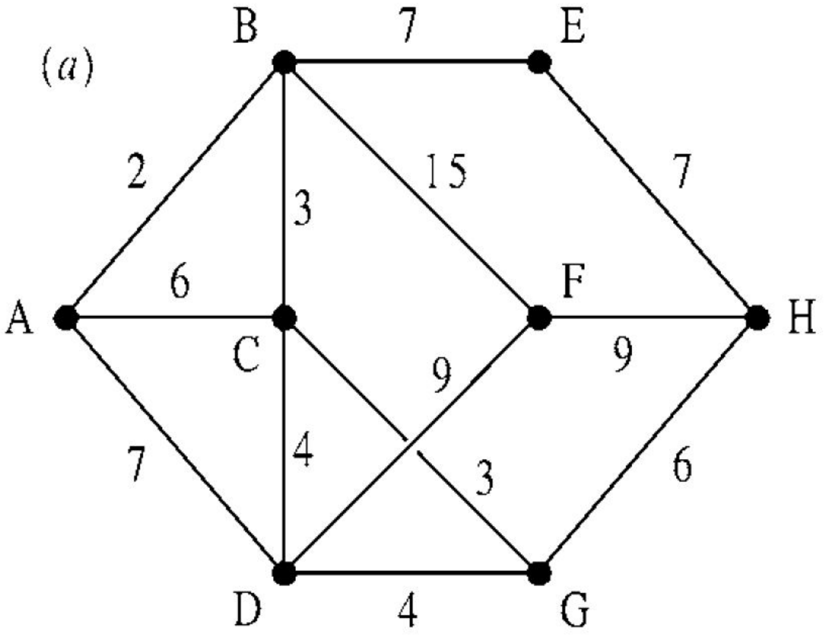


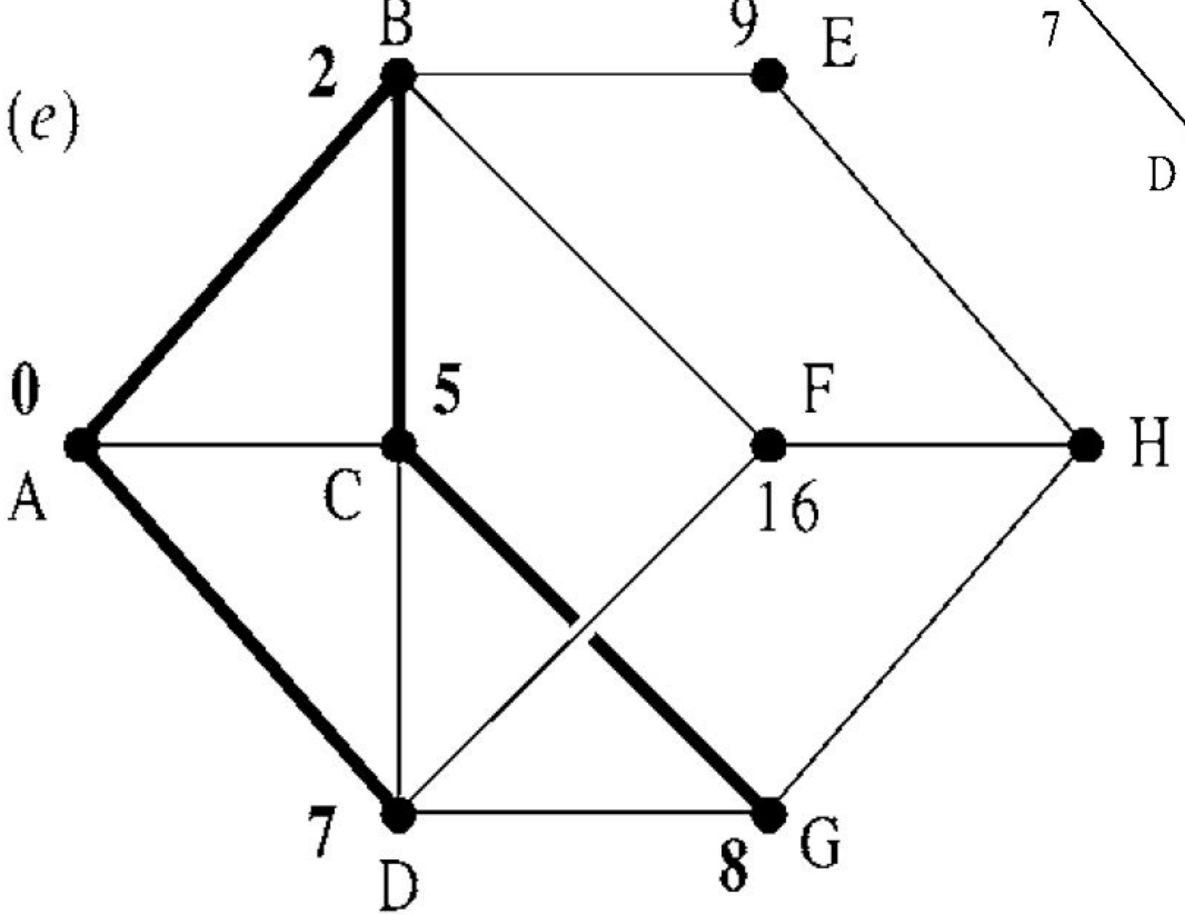

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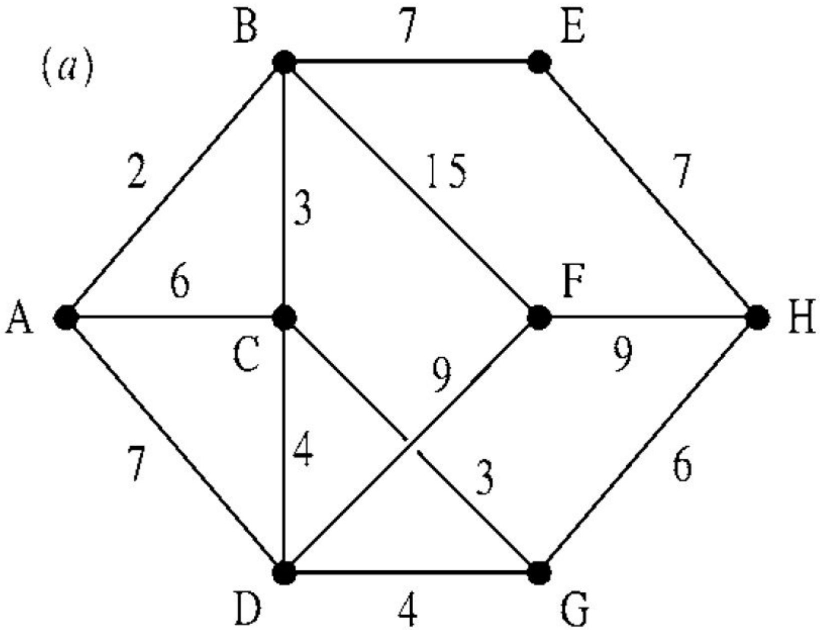


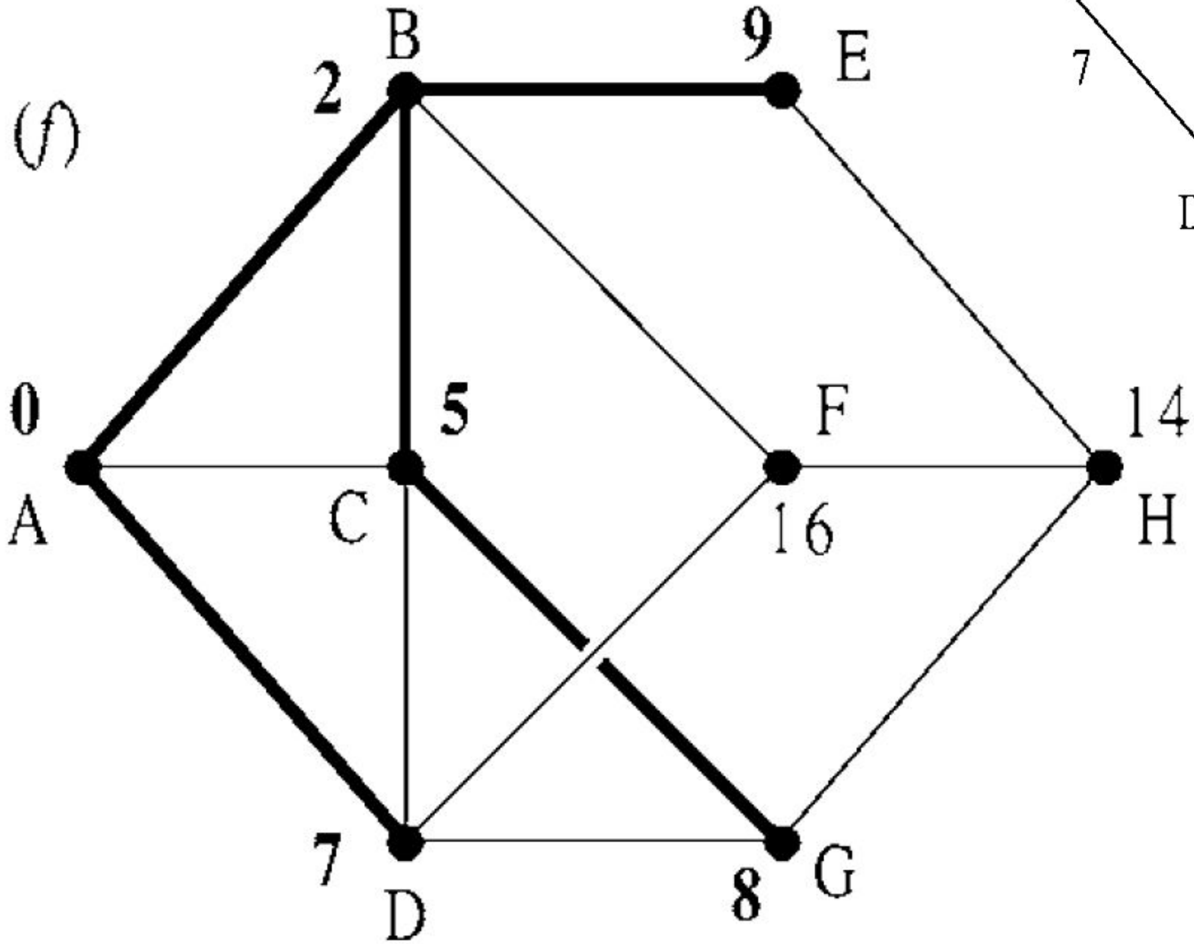

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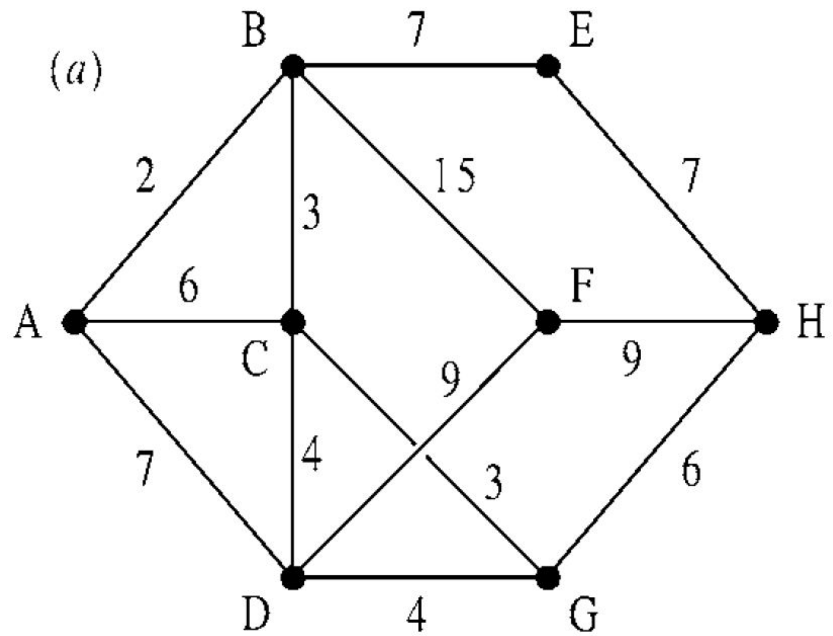
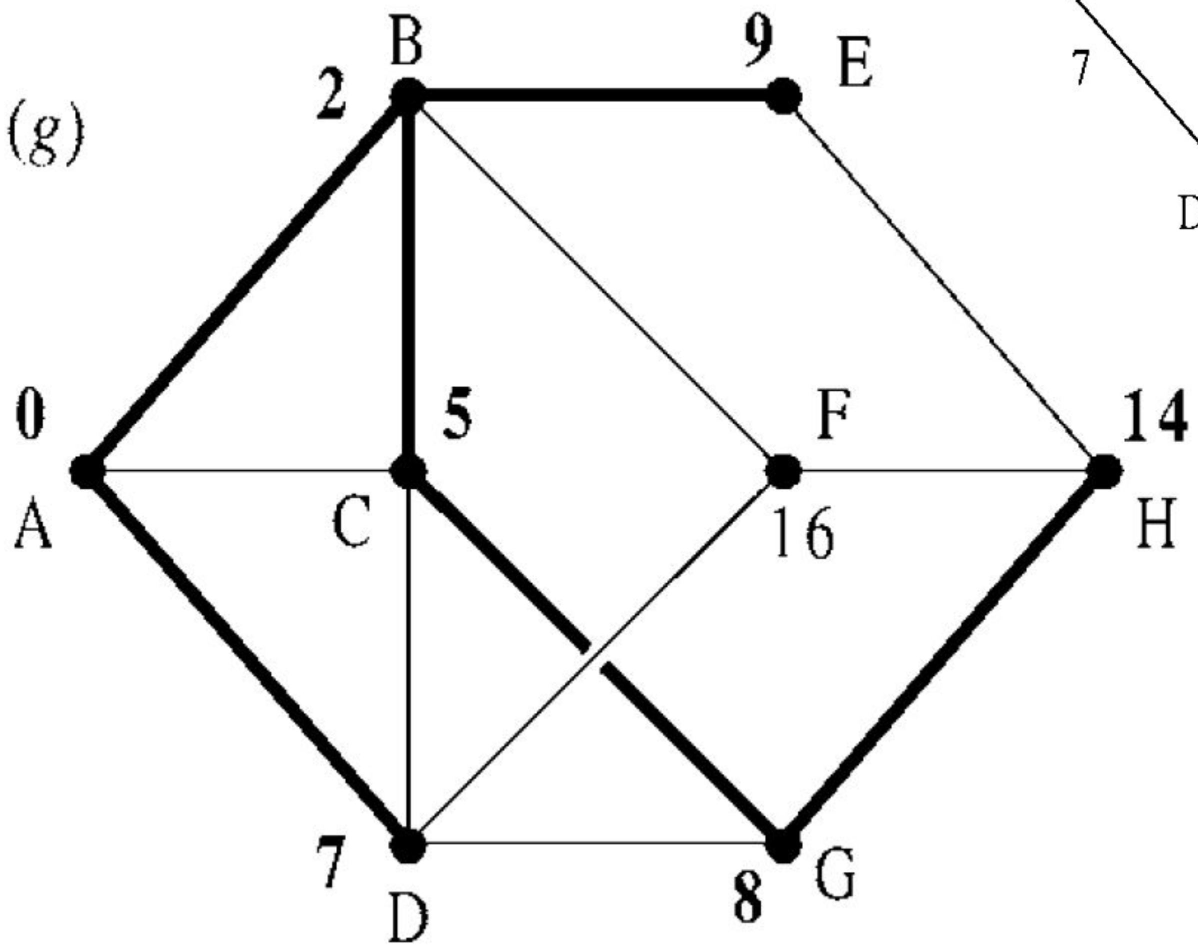
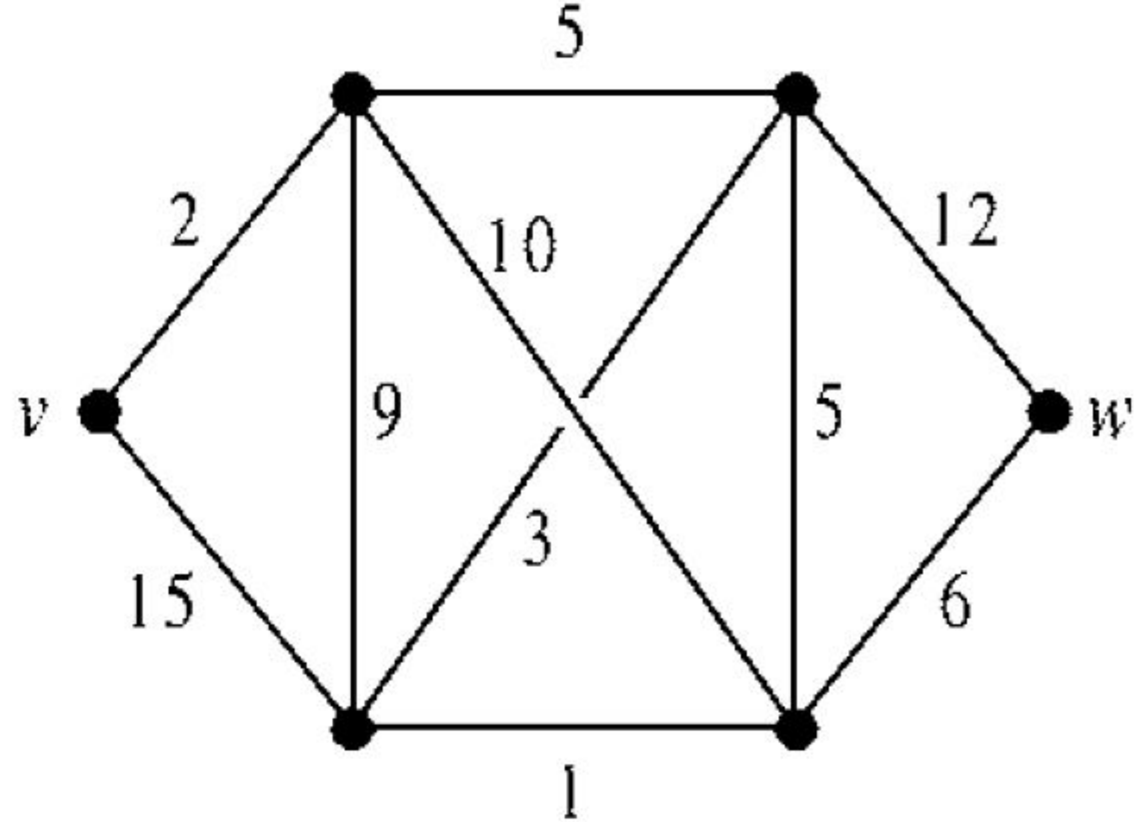


# Exercise 2

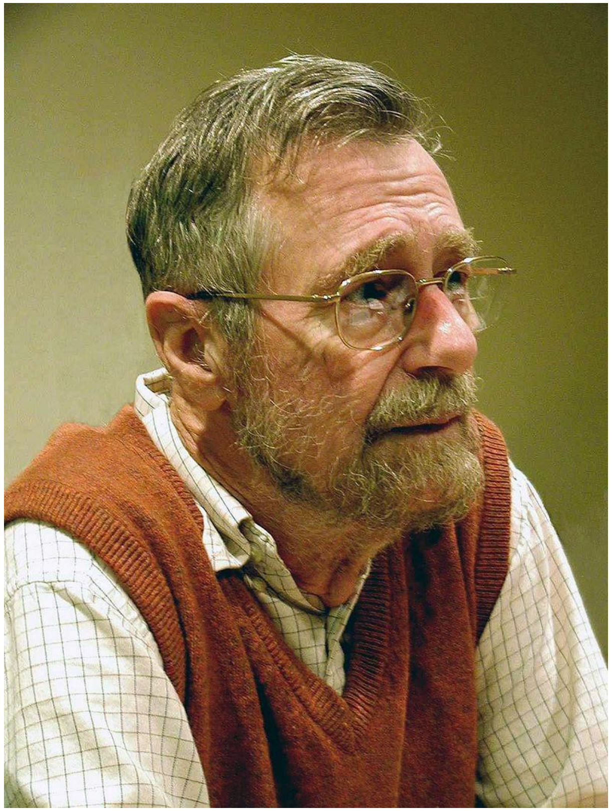
- Apply Dijkstra’s algorithm to obtain the shortest path from $v$ to $w$ in the following weighted graph.



# Biography

***Edsger Dijkstra, born in the Netherlands, began programming computers in the early 1950s while studying theoretical physics at the University of Leiden.***

***In 1952, realizing that he was more interested in programming than in physics, he quickly completed the equirements for his physics degree and began his career as a programmer, even though programming was not a recognized profession.***

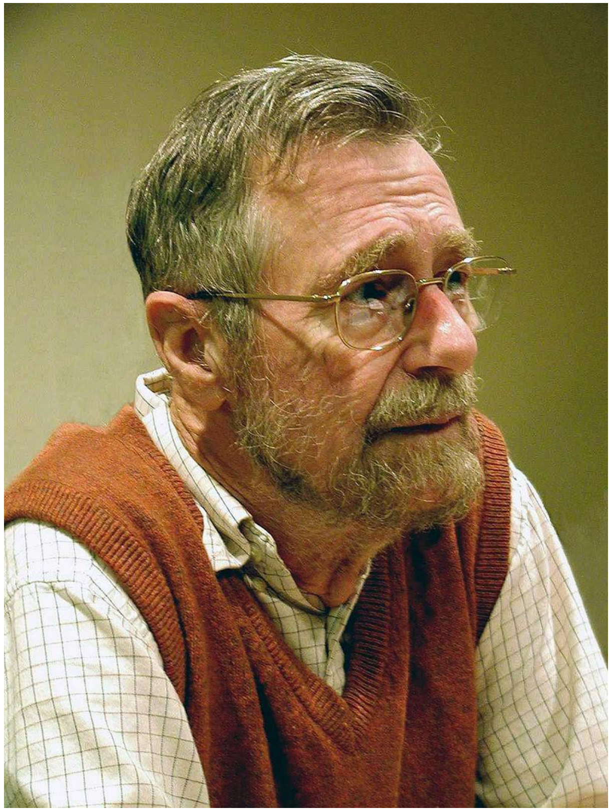


**Edsger Wybe Dijkstra**
**1930 – 2002**

# Biography

***In 1957, the authorities in Amsterdam refused to accept "programming" as his profession on his marriage license. However, they did accept "theoretical physicist" when he changed his entry to this.***

***Dijkstra was one of the most forceful proponents of programming as a scientific discipline.***



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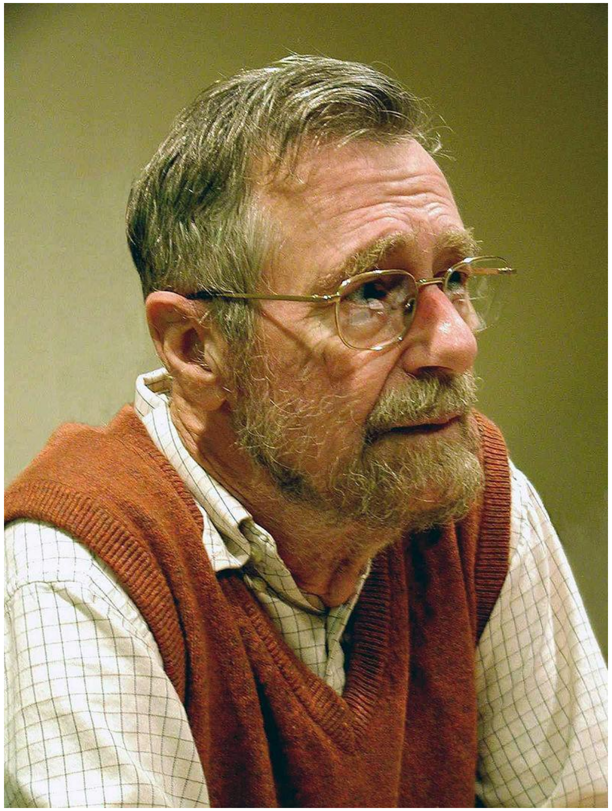
# Biography

***In 1972 Dijkstra received the Turing Award from the Association for Computing Machinery, one of the most prestigious awards in computer science.***

***Dijkstra became a Burroughs Research Fellow in 1973, and in 1984 he was appointed to a chair in Computer Science at the University of Texas, Austin.***



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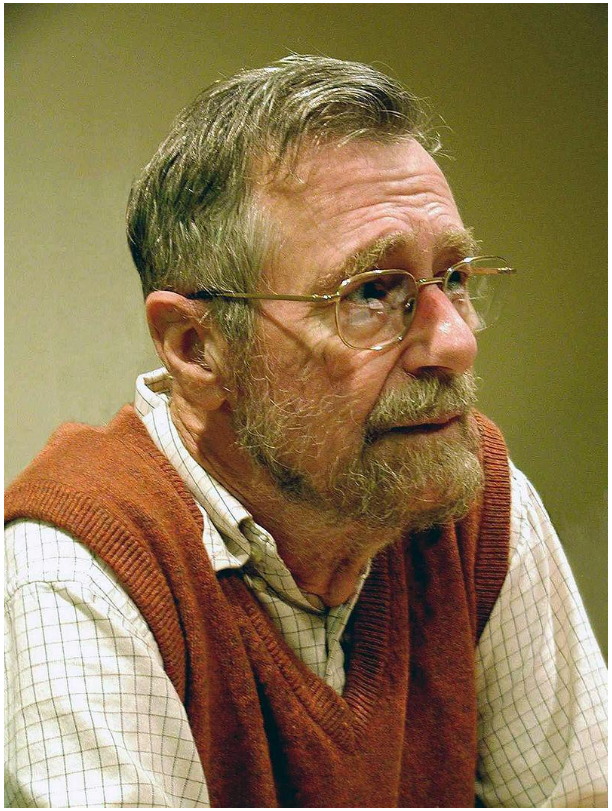
# Biography

***He has made fundamental contributions to the areas of operating systems, including deadlock avoidance; programming languages, including the notion of structured programming; and algorithms.***



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