Determinants

1 The Determinant of a Matrix

2 Properties of Determinants

3 Application of Determinants: Cramer's Rule

1 The Determinant of a Matrix

Determinant - a square array of numbers or variables enclosed between parallel vertical bars.

To find a determinant you must have a ***SQUARE MATRIX*!!**

Finding a 2 x 2 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Given a **square matrix** A its **determinant** is a real number associated with the matrix.
- The determinant of A is written:

$$\det(A) \qquad \text{or} \qquad |A|$$

- For a 2x2 matrix, the definition is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- For larger matrices the definition is more complicated

※ The determinant is NOT a matrix operation

※ The determinant is a kind of information extracted from a square matrix to reflect some characteristics of that square matrix

※ For example, this chapter will discuss that matrices with a zero determinant are with very different characteristics from those with non-zero determinants

※ The motives to calculate determinants are to identify the characteristics of matrices and thus facilitate the comparison between matrices since it is impossible to investigate or compare matrices entry by entry

※ The similar idea is to compare groups of numbers through the calculation of averages and standard deviations

※ Not only the determinant but also the eigenvalues and eigenvectors are the information that can be used to identify the characteristics of square matrices

- The determinant of a 2 × 2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Note:

1. For every SQUARE matrix, there is a real number associated with this matrix and called its *determinant*
2. It is common practice to omit the matrix brackets

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- Historically speaking, the use of determinants arose from the recognition of special patterns that occur in the solutions of linear systems:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{cases}$$

$$\Rightarrow \; x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \text{ and } x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

- Note:

1. x_1 and x_2 have the same denominator, and this quantity is called the determinant of the coefficient matrix A

2. There is a unique solution if $a_{11}a_{22} - a_{21}a_{12} = |A| \neq 0$

Determinants 2x2 examples

$$\det\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix} = (1)(4) - (2)(3) = -2$$

$$\det\begin{pmatrix}-5 & 2\\ -2 & 0\end{pmatrix} = \begin{vmatrix}-5 & 2\\ -2 & 0\end{vmatrix} = (-5)(0) - (2)(-2) =$$

4

$$\det\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix} = \begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix} = (1)(4) - (2)(2) = 0$$

- Note: The determinant of a matrix can be positive, zero, or negative

Determinants

- To define det($\underline{A}$) for larger matrices, we will need the definition of a **minor** $\underline{M}_{ij}$
- The minor $\underline{M}_{ij}$ of a matrix $\underline{A}$ is the matrix formed by removing the ^{i}th row and the ^{j}th column of $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{11}$: remove row 1, col 1

$$\underline{M}_{11} = \begin{pmatrix} 2 & 3 \\ 7 & 0 \end{pmatrix}$$

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$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{12}$: remove row 1, col 2

$$\underline{M}_{12} = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$$

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$$\underline{M}_{13} = \begin{pmatrix} -1 & 2 \\ 2 & 7 \end{pmatrix}$$

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$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{21}$: remove row 2, col 1

$$\underline{M}_{21} = \begin{pmatrix} 1 & -2 \\ 7 & 0 \end{pmatrix}$$

Determinants

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- The minor $\underline{M}_{ij}$ of a matrix $\underline{A}$ is the matrix formed by removing the ^{i}th row and the ^{j}th column of $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{22}$: remove row 2, col 2

$$\underline{M}_{22} = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$$

Determinants

- To define det($\underline{A}$) for larger matrices, we will need the definition of a **minor** $\underline{M}_{ij}$
- The minor $\underline{M}_{ij}$ of a matrix $\underline{A}$ is the matrix formed by removing the ith row and the jth column of $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{23}$: remove row 2, col 3

$$\underline{M}_{23} = \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix}$$

Determinants

- To define det($\underline{A}$) for larger matrices, we will need the definition of a **minor** $\underline{M}_{ij}$
- The minor $\underline{M}_{ij}$ of a matrix $\underline{A}$ is the matrix formed by removing the ^{i}th row and the ^{j}th column of $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{31}$: remove row 3, col 1

$$\underline{M}_{31} = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$$

Determinants

- To define det($\underline{A}$) for larger matrices, we will need the definition of a **minor** $\underline{M}_{ij}$
- The minor $\underline{M}_{ij}$ of a matrix $\underline{A}$ is the matrix formed by removing the ^{i}th row and the ^{j}th column of $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{32}$: remove row 3, col 2

$$\underline{M}_{32} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

Determinants

- To define det($\underline{A}$) for larger matrices, we will need the definition of a **minor** $\underline{M}_{ij}$
- The minor $\underline{M}_{ij}$ of a matrix $\underline{A}$ is the matrix formed by removing the ^{i}th row and the ^{j}th column of $\underline{A}$

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$\underline{M}_{33}$: remove row 3, col 3

$$\underline{M}_{33} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

The formula for a 3x3 matrix

- For a matrix

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Its determinant is given by

$$|\underline{A}| = a_{11}|\underline{M}_{11}| - a_{12}|\underline{M}_{12}| + a_{13}|\underline{M}_{13}|$$

- From the formula for a 2x2 matrix:

$$|\underline{M}_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

The formula for a 3x3 matrix

- For a matrix

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Its determinant is given by

$$|\underline{A}| = a_{11}|\underline{M}_{11}| - a_{12}|\underline{M}_{12}| + a_{13}|\underline{M}_{13}|$$

- From the formula for a 2x2 matrix:

$$|\underline{M}_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

The formula for a 3x3 matrix

- For a matrix

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Its determinant is given by

$$|\underline{A}| = a_{11}|\underline{M}_{11}| - a_{12}|\underline{M}_{12}| + a_{13}|\underline{M}_{13}|$$

- From the formula for a 2x2 matrix:

$$|\underline{M}_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

3x3 Example

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 3 \\ 2 & 7 & 0 \end{pmatrix}$$

$$|\underline{A}| = 1x|\underline{M}_{11}| - 1x|\underline{M}_{12}| + (-2)x|\underline{M}_{13}|$$

$$|\underline{A}| = 1x \begin{vmatrix} 2 & 3 \\ 7 & 0 \end{vmatrix} - 1x \begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} + (-2) \begin{vmatrix} -1 & 2 \\ 2 & 7 \end{vmatrix}$$

$$= 1x(-21) -1x(-6) +(-2)x(-11) = 7$$

3x3 Example

$$\underline{B} = \begin{pmatrix} 0 & 1 & 3 \\ 5 & 3 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

$$|\underline{B}| = 0x|\underline{M}_{11}| - 1x|\underline{M}_{12}| + 3x|\underline{M}_{13}|$$

$$|\underline{B}| = 0x \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} - 1x \begin{vmatrix} 5 & 1 \\ -1 & 0 \end{vmatrix} + 3x \begin{vmatrix} 5 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= 0x(-2) - 1x(1) + (3)x(13) = 38$$

The formula for a 3x3 matrix

- For the matrix

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- We used the top row to calculate the determinant:

$$|\underline{A}| = a_{11}|\underline{M}_{11}| - a_{12}|\underline{M}_{12}| + a_{13}|\underline{M}_{13}|$$

- However, we could equally have used **any row** of the matrix and performed a similar calculation

The formula for a 3x3 matrix

- For the matrix

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- Using the **top** row:

$$|\underline{A}| = a_{11}|\underline{M}_{11}| - a_{12}|\underline{M}_{12}| + a_{13}|\underline{M}_{13}|$$

- Using the **second** row

$$|\underline{A}| = -a_{21}|\underline{M}_{21}| + a_{22}|\underline{M}_{22}| - a_{23}|\underline{M}_{23}|$$

- Using the **third** row

$$|\underline{A}| = a_{31}|\underline{M}_{31}| - a_{32}|\underline{M}_{32}| + a_{33}|\underline{M}_{33}|$$

The formula for a 3x3 matrix

$$|\underline{A}| = a_{11}|\underline{M}_{11}| - a_{12}|\underline{M}_{12}| + a_{13}|\underline{M}_{13}|$$

$$= -a_{21}|\underline{M}_{21}| + a_{22}|\underline{M}_{22}| - a_{23}|\underline{M}_{23}|$$

$$= a_{31}|\underline{M}_{31}| - a_{32}|\underline{M}_{32}| + a_{33}|\underline{M}_{33}|$$

- Notice the **changing signs** depending on what row we use:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

The formula for a 3x3 matrix

- Equally, we could have used any **column** as long as we follow the **signs** pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- E.g. using the first **column**:

$$|\underline{A}| = a_{11}|\underline{M}_{11}| - a_{21}|\underline{M}_{21}| + a_{31}|\underline{M}_{31}|$$

- This choice sometimes makes it a bit easier to calculate determinants. e.g.

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

- Using the first row:

$$|\underline{A}| = 1\text{x} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 1\text{x} \begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} + (-2)\text{ x} \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 1\text{x}(-1) -1\text{x}(0) + (-2)\text{x}(0) = -1$$

- This choice sometimes makes it a bit easier to calculate determinants. e.g.

$$\underline{A} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

- However, using the first **column**:

$$|\underline{A}| = 1x \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - 0 + 0 = 1x(-1) = -1$$

A general formula for determinants

- For a 4x4 matrix we add up minors like the 3x3 case, and again use the same signs pattern

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

- Notice that if we think of the signs pattern as a matrix, then it can be written as $(-1)^{i+j}$

A general formula for determinants

Calculation of Determinants

Methods of determinant calculation are based on the properties of determinants. Here we consider two methods which being combined together result in the most efficient computing technique.

Expanding a determinant by a row or column

Before formulating the theorem, let us introduce a few definitions.
Let A be a square matrix of the order n. By removing the i-th row and j-th column, we obtain a submatrix of A, having the order $(n-1)$. The determinant of that submatrix is called the minor of the element $a_{i,j}$, which is denoted by $M_{i,j}$.

- Minor of the entry a_{ij}: the determinant of the matrix obtained by deleting the *i*-th row and *j*-th column of *A*

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

The **cofactor** of the element $a_{i,j}$ is defined as the minor $M_{i,j}$ with the sign $(-1)^{i+j}$. It is denoted by the symbol $A_{i,j}$:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

- Ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow A_{21} = (-1)^{2+1} M_{21} = -M_{21} \qquad A_{22} = (-1)^{2+2} M_{22} = M_{22}$$

- Notes: Sign pattern for cofactors. Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs. (Positive and negative signs appear alternately at neighboring positions.)

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Theorem: Expansion by cofactors

Let A be a square matrix of order n, then the determinant of A is given by

(a) $$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}A_{ij} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

(cofactor expansion along the i-th row, i=1, 2,…, n)

or

(b) $$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}A_{ij} = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

(cofactor expansion along the j-th column, j=1, 2,…, n)

※The determinant can be derived by performing the cofactor expansion along any row or column of the examined matrix

- Ex: The determinant of a square matrix of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad \text{(first row expansion)} \\ &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \quad \text{(second row expansion)} \\ &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \quad \text{(third row expansion)} \\ &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \quad \text{(first column expansion)} \\ &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \quad \text{(second column expansion)} \\ &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \quad \text{(third column expansion)} \end{aligned}$$

- Ex: The determinant of a square matrix of order 3

$$A=\begin{bmatrix}0 & 2 & 1\\ 3 & -1 & 2\\ 4 & 0 & 1\end{bmatrix} \quad \Rightarrow \det(A)=?$$

Sol:

$$A_{11}=(-1)^{1+1}\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=-1 \qquad A_{12}=(-1)^{1+2}\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}=(-1)(-5)=5$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix}3 & -1\\ 4 & 0\end{vmatrix}=4$$

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}\\ &=(0)(-1)+(2)(5)+(1)(4)\\ &=14\end{aligned}$$

- Ex: The determinant of a square matrix of order 4

$$A=\begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A)=?$$

Sol:

$$\det(A) = (3)(A_{13}) + (0)(A_{23}) + (0)(A_{33}) + (0)(A_{43}) = 3A_{13}$$

$$= 3(-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3\left[(0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}\right]$$

$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

$$= (3)(13)$$

$$= 39$$

※ By comparing the exercises, it is apparent that the computational effort for the determinant of 4×4 matrices is much higher than that of 3×3 matrices.

- Upper triangular matrix:

  All entries below the main diagonal are zeros

- Lower triangular matrix:

  All entries above the main diagonal are zeros

- Diagonal matrix:

  All entries above and below the main diagonal are zeros

Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular   lower triangular   diagonal

- Theorem: (Determinant of a Triangular Matrix)

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdots a_{nn}$$

- Ex: Find the determinants of the following triangular matrices

$$\text{(a)} \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix} \qquad \text{(b)} \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$\text{(a)} \quad |A| = (2)(-2)(1)(3) = -12$$

$$\text{(b)} \quad |B| = (-1)(3)(2)(4)(-2) = 48$$

2 Properties of Determinants

- Conditions that yield a zero determinant

  If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$

  (a) An entire row (or an entire column) consists of zeros

  (b) Two rows (or two columns) are equal

  (c) One row (or column) is a multiple of another row (or column)

- Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Theorem: Determinant of a matrix product

$$\det(AB) = \det(A)\det(B)$$

(Verified by Ex on the next slide)

- Notes:

(1) $\det(A_1 A_2 \cdots A_n) = \det(A_1)\det(A_2)\cdots\det(A_n)$

(2) $\det(A+B) \neq \det(A) + \det(B)$

(3) $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Ex 1: The determinant of a matrix product

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7 \qquad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow \quad |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

- Check:

$$|AB| = |A|\ |B|$$

- Theorem: Determinant of a scalar multiple of a matrix

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

(can be proven by repeatedly use the fact that if $B = M_i^{(k)}(A) \Rightarrow |B| = k|A|$)

- Ex 2:

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \text{ if } \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5, \text{ find } |A|$$

Sol:

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

Theorem: (Determinant of an invertible matrix)

A square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$

▪ Ex 3: Classifying square matrices as singular or nonsingular

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \quad \Rightarrow \quad$ A has no inverse (it is singular)

$|B| = -12 \neq 0 \quad \Rightarrow \quad$ B has inverse (it is nonsingular)

Inverse Matrices

Let A be a square matrix.

A matrix A^{-1} is called an **inverse matrix** of A if

$$A^{-1}A = AA^{-1} = I,$$

where I is an identity matrix.

If the determinant of a matrix is equal to zero, then the matrix is called **singular**; otherwise, if $\det A \neq 0$, the matrix A is called **regular**.

If each element of a square matrix A is replaced by its cofactor, then the transpose of the matrix obtained is called the **adjoint matrix** of A:

$$\operatorname{adj} A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{pmatrix}^T = \begin{pmatrix} A_{1,1} & A_{2,1} & \cdots & A_{n,1} \\ A_{1,2} & A_{2,2} & \cdots & A_{n,2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1,n} & A_{2,n} & \cdots & A_{n,n} \end{pmatrix}.$$

Theorem of Inverse Matrices

For any regular matrix A there exists the unique inverse matrix:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A \,.$$

Any singular matrix has no an inverse matrix.

Examples of Calculations of Inverse Matrices

Example 1: Given the matrix $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$, find the inverse of A.

Solution: First, calculate the determinant:

$$\det A = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2.$$

Next, find the cofactors of all elements:

$$A_{1,1} = (-1)^{1+1} 2 = 2, \qquad A_{1,2} = (-1)^{1+2} \cdot 1 = -1,$$

$$A_{2,1} = (-1)^{2+1} \cdot 4 = -4, \qquad A_{2,2} = (-1)^{2+2} 3 = 3.$$

Then, find the adjoint matrix of A:

$$\operatorname{adj} A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}^T = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}.$$

Finally, obtain

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

Verification:

$$AA^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$

and

$$A^{-1}A = \frac{1}{2}\begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Example 2: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Find the inverse of A.

Solution: Calculate the determinant:

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0.$$

Therefore, the given matrix is singular, and so it has no the inverse of A.

Example 3

$$\text{Let } A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}.$$

Solve for X the matrix equation

$$XA = B.$$

Solution: Note that $\det A = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$, that is, A is a regular matrix.

Therefore, there exists the inverse of A:

$$X = B \cdot A^{-1}.$$

Find the inverse of matrix A.

$$\operatorname{adj} A = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \qquad \Rightarrow$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}.$$

Thus,

$$X = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -17 \\ 5 & -12 \end{pmatrix}.$$

Verification:

$$X \cdot A = \begin{pmatrix} 7 & -17 \\ 5 & -12 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \equiv B \,.$$

- Theorem: Determinant of an inverse matrix

$$\text{If } A \text{ is invertible, then } \det(A^{-1}) = \frac{1}{\det(A)}$$

- Theorem: Determinant of a transpose

$$\text{If } A \text{ is a square matrix, then } \det(A^{T}) = \det(A)$$

- Ex 4:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad \text{(a) } \left|A^{-1}\right| = ? \qquad \text{(b) } \left|A^{T}\right| = ?$$

Sol:

$$\because \quad |A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4$$

$$\therefore \quad \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

$$\left|A^{T}\right| = |A| = 4$$

- The similarity between the noninvertible matrix and the real number 0

	Matrix A	**Real number c**
Invertible	$\det(A) \neq 0$ A^{-1} exists and $\det(A^{-1}) = \dfrac{1}{\det(A)}$	$c \neq 0$ c^{-1} exists and $c^{-1} = \dfrac{1}{c}$
Noninvertible	$\det(A) = 0$ A^{-1} does not exist $\left(\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{0}\right)$	$c = 0$ c^{-1} does not exist $\left(c^{-1} = \dfrac{1}{c} = \dfrac{1}{0}\right)$

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent

(1) A is invertible

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(4) $\det(A) \neq 0$

- Ex 5: Which of the following system has a unique solution?

(a)
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

(b)
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

Sol:

(a) $A\mathbf{x} = \mathbf{b}$ (the coefficient matrix is the matrix A in Ex 3)

$\because$ $|A| = 0$ (from Ex 3)

$\therefore$ This system does not have a unique solution

(b) $B\mathbf{x} = \mathbf{b}$ (the coefficient matrix is the matrix B in Ex 3)

$\because$ $|B| = -12 \neq 0$ (from Ex 3)

$\therefore$ This system has a unique solution

3 Applications of Determinants

- Theorem: Cramer's Rule

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \Rightarrow A\mathbf{x} = \mathbf{b}$$

$A^{(i)}$ represents the i-th column vector in A

where $A = \left[a_{ij}\right]_{n\times n} = \left[A^{(1)}\ A^{(2)}\ \cdots\ A^{(n)}\right]$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Suppose this system has a unique solution, i.e.,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

By defining $A_j = \begin{bmatrix} A^{(1)} & A^{(2)} & \cdots & A^{(j-1)} & \mathbf{b} & A^{(j+1)} & \cdots & A^{(n)} \end{bmatrix}$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

(i.e., $\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$)

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \; j = 1, 2, \cdots, n$$

- Ex: Use Cramer's rule to solve the system of linear equation

$$\begin{aligned} -x &+ 2y - 3z = 1 \\ 2x &\qquad\;\; + z = 0 \\ 3x &- 4y + 4z = 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \qquad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

Keywords

- determinant
- minor
- cofactor
- expansion by cofactors
- upper triangular matrix
- lower triangular matrix
- diagonal matrix
- Cramer’s rule