Mathematics for Computing

Lecture 2:
Logarithms and indices

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Material

- What are Logarithms?
- Laws of indices
- Logarithmic identities

Exponents

- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 2 \times 2 = 4$
- $2^3 = 2 \times 2 \times 2 = 8$,

…

- $2^n = 2 \times 2 \times \ldots$ with n 2s

- $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
- $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
- $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

…

- $2^{-n} = \frac{1}{2^n} = \frac{1}{2x2x\ldots with\ n\ 2s}$

Problem

- We want to know how many bits the number 456 will require when stored in (non signed) binary format.
- Solution based on what we learned last week: Convert the number to Binary and count the number of bits
- After counting we get 9 (check it out)
- There is a simpler way

Digit number	Number	Remainder when dividing by 2
1	456	0
2	228	0
3	114	0
4	57	1
5	28	0
6	14	1
7	7	1
8	3	1
9	1	1

A simpler way

- Round 456 up to the smallest power of 2 that is greater than 456.
- Specifically, 512.
- Notice that $512 = 2^9$.
- Why did we round up?

The answer!

index	9	8	7	6	5	4	3	2	1	0
456		1	1	1	1	0	1	0	0	0
512	1	0	0	0	0	0	0	0	0	0
	2^9									

This gives us 2 to the power of the 1 + the index of the MSB of our number, which is 1 less than its number of bits because the indices start from 0!

A simpler way

- Much better, but *we really don't like* the rounding up to the smallest …
- Don't worry we just did this specific rounding up so that the answer comes out nicely.
- We will show a simpler way to do this (although we will start with 512 since it is nicer)

Logarithms

- If we already knew the 512, then we would wonder which number is such that

$$2^x = 512$$

- In words, how many times do we need to multiply 2 by itself to get 512?
- The formal way to write this is $x = \log_2 512$, which means how many times do we need to multiply 2 by itself to get 512?
- We already know the answer is 9.
- This is interpreted as follows: $2^{\log_2 512} = 2^9 = 512$

Logarithms

- We only know 456, lets compute log base 2 of 456

$$\log_2 456 = 8.861\ldots$$

- Rounding this number up gives the answer we wanted, 9!
- Why didn't we get an integer? Because 456 is not a power of 2 so to get 456 we need to multiply 2 by itself 8.861 times, which can be done once we know what this means.
- So, how many bits do need in order to store the number 3452345 in binary format?

Logarithms

- If $x = y^z$
- then $z = \log_y x$

Logarithms and Exponents

- If $x = y^z$
- then $z = \log_y x$

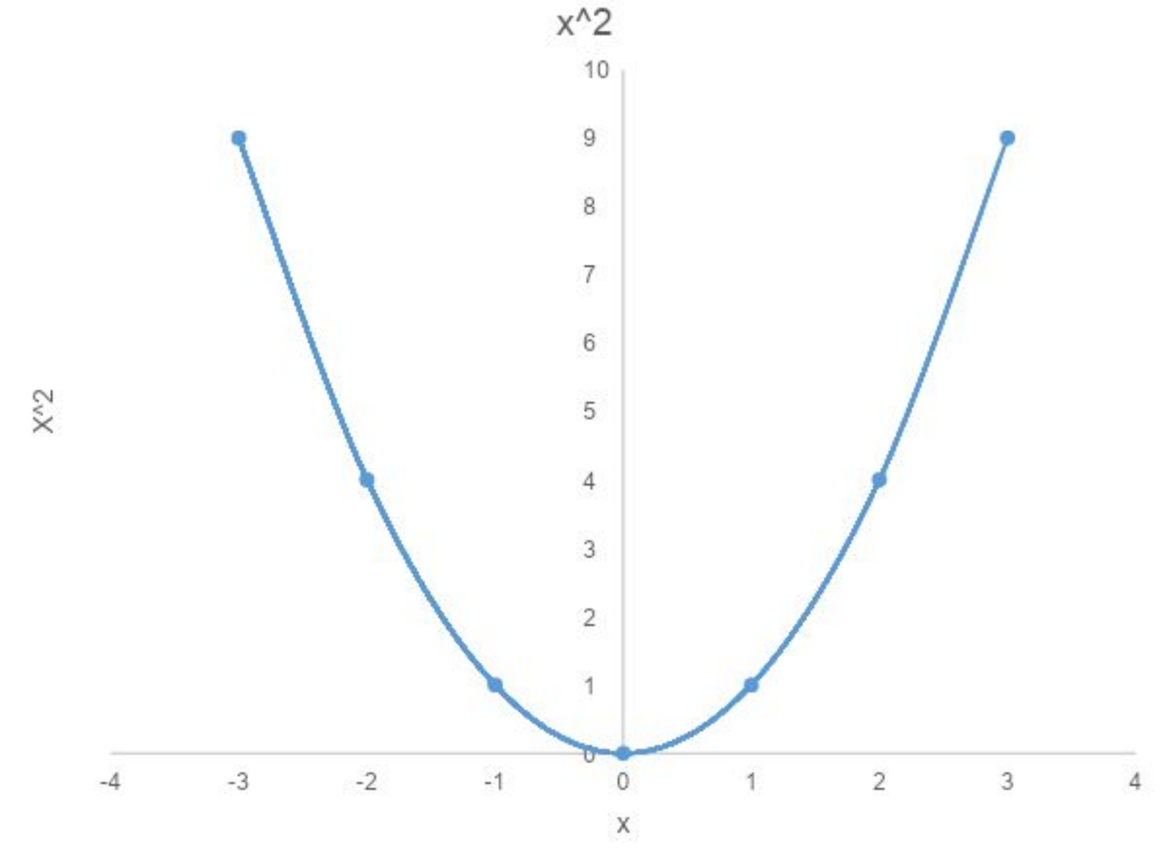
The base

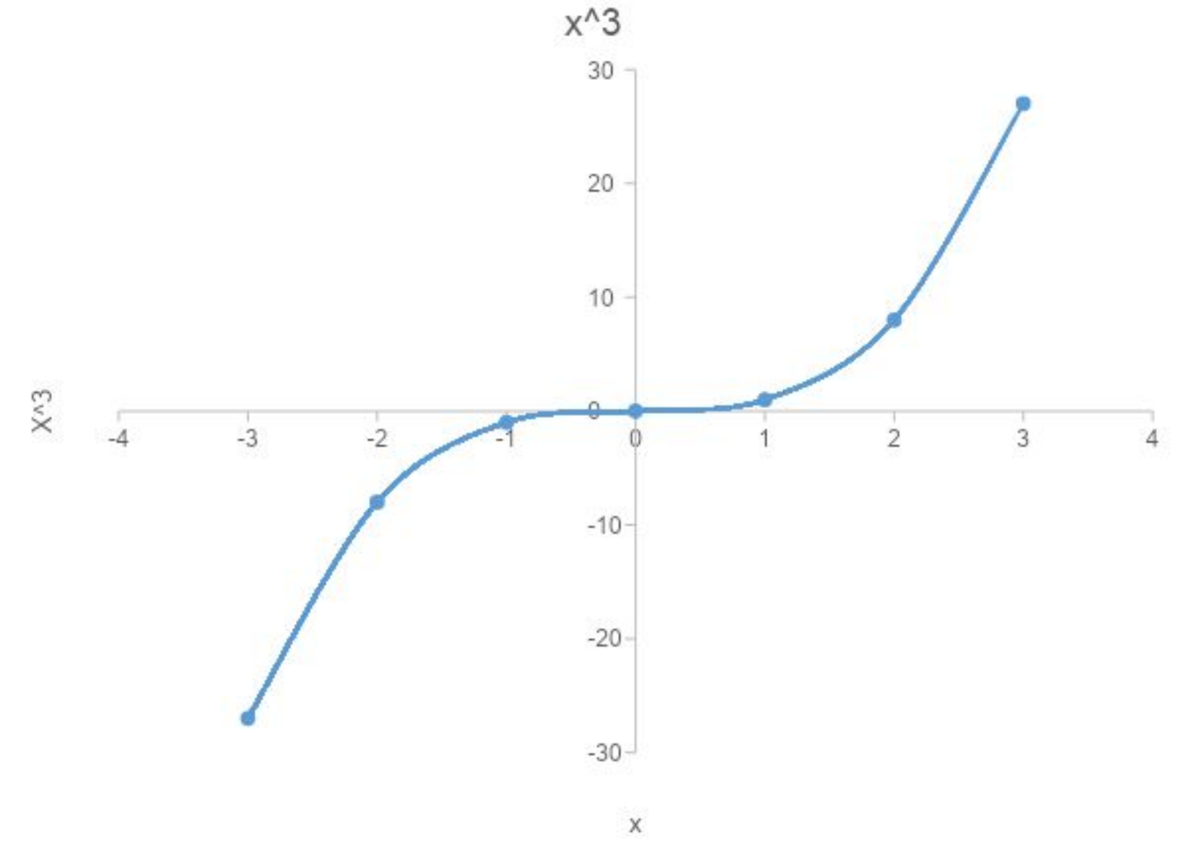
- e.g. $1000 = 10^3$,
- then $3 = \log_{10} (1000)$

Logarithms and Exponents: general form

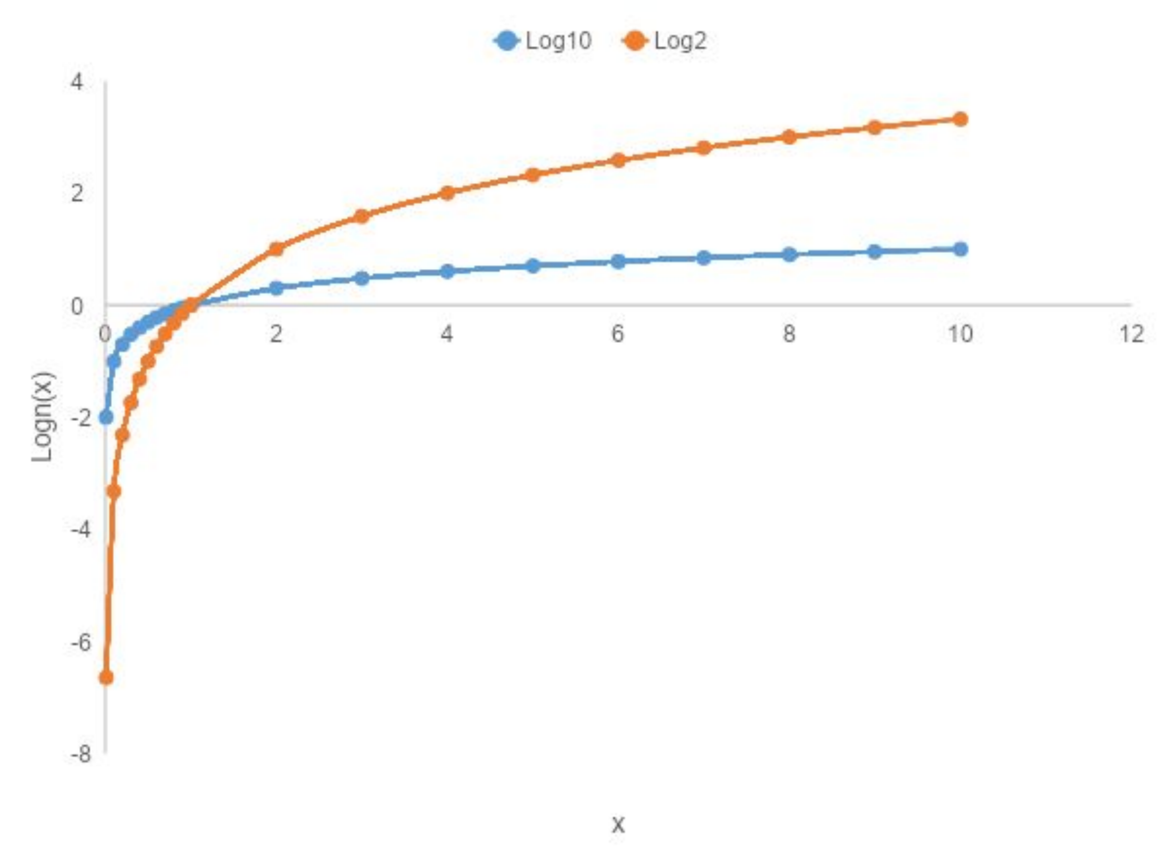
- From lecture 1) base index form:
  number = $\text{base}^{\text{index}}$
- then index = $\log_{\text{base}}$ (number)

Graphs of exponents

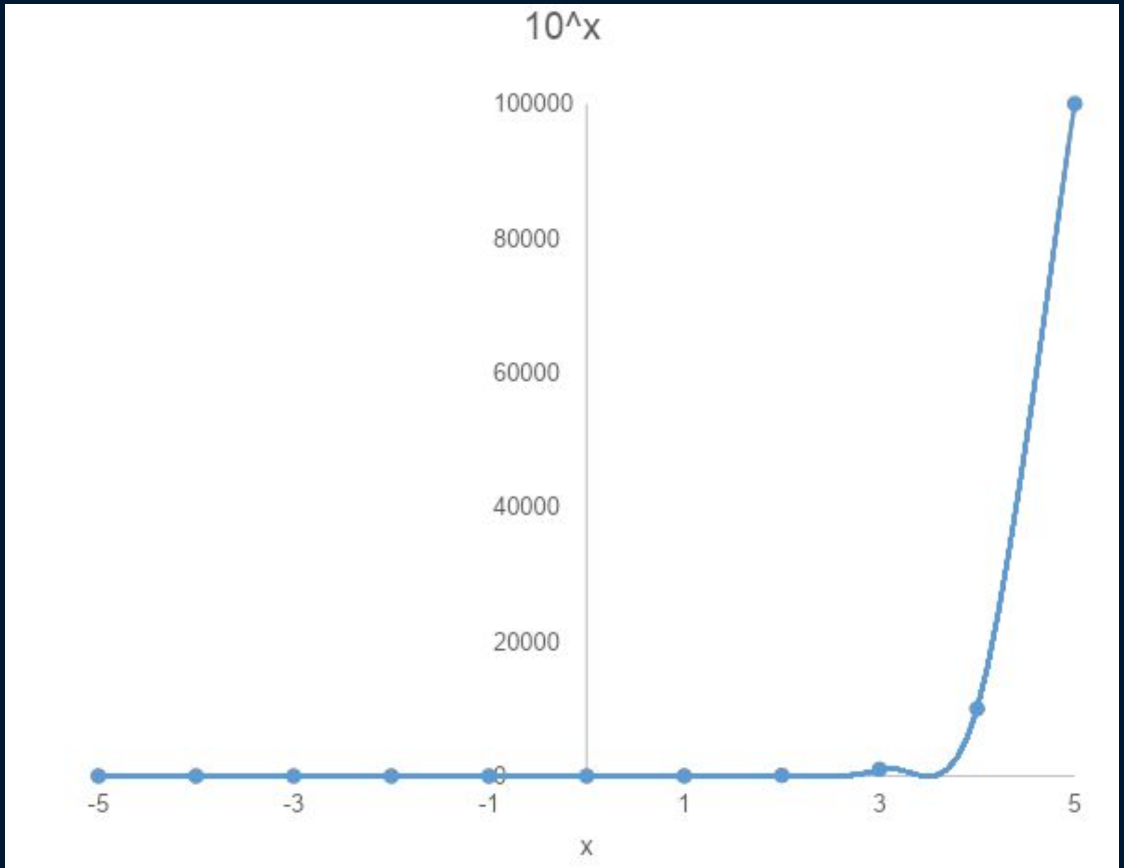




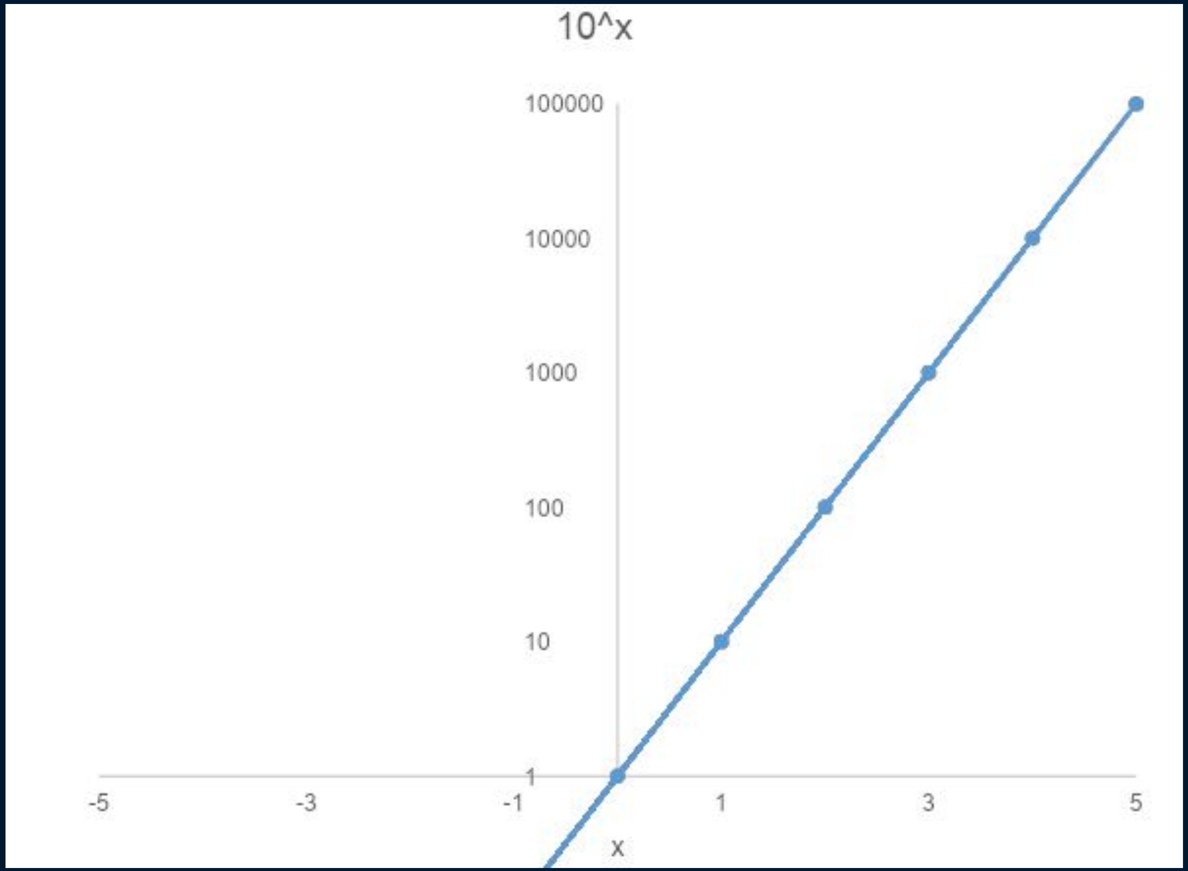
Graphs of logarithms



Log plot



10^x
100000
80000
60000
40000
20000
0
-5
-3
-1
1
3
5
x



10^x
100000
10000
1000
100
10
1
-5
-3
-1
1
3
5
x

Three ‘special’ types of logarithms

- Common Logarithm: base 10
  Common in science and engineering
- Natural Logarithm: base e (≈2.718).
  Common in mathematics and physics
- Binary Logarithm: base 2
  Common in computer science

Laws of indices

1) $a^0 = 1$
2) $a^1 = a$

Laws of indices

1) $a^0 = 1$
2) $a^1 = a$

Examples:

- $2^0 = 1$
- $10^0 = 1$

Laws of indices

1) $a^0 = 1$

2) $a^1 = a$

Examples:

- $2^1 = 2$
- $10^1 = 10$

Laws of indices

3) $a^{-x} = 1/a^x$

Laws of indices

3) $a^{-x} = 1/a^x$

Example:

- $3^{-2} = 1/3^2 = 1/27$

Laws of indices

4) $a^x \cdot a^y = a^{(x + y)}$

(a multiplied by itself x times) · (a multiplied by itself y times) = **a** multiplied by itself $x+y$ times

5) $a^x / a^y = a^{(x - y)}$

(a multiplied by itself x times) divided by (a multiplied by itself y times) = **a** multiplied by itself $x-y$ times

Laws of indices

4) $a^x \cdot a^y = a^{(x + y)}$

- $4^2 \cdot 4^3 = 4^{(2+3)} = 4^5$
  $16x64 = 1024$
  $9 \cdot 27 = 3^2 \cdot 3^3 = 3^{(3 + 2)} = 3^5 = 243$
  $25 \cdot (1/5) = 5^2 \cdot 5^{-1} = 5^{(2-1)} = 5^1 = 5$

Laws of indices

5) $a^x / a^y = a^{(x - y)}$

$10^5 / 10^3 = 10^{(5-3)} = 10^2$

- 100,000 / 1,000 = 100

$2^3 / 2^7 = 2^{(3-7)} = 2^{-4}$

8 / 128 = 1/16, [2^4 = 16, 2^{-4} = 1/16, see law 3)]

64 / 4 = $2^6 / 2^2 = 2^{(6-2)} = 2^4 = 16$

27 / 243 = $3^3 / 3^5 = 3^{(3-5)} = 3^{-2} = 1/9$

25 / (1/5) = $5^2 / 5^{-1} = 5^{(2+1)} = 5^3 = 125$

Laws of indices

- 6) $(a^x)^y = a^{xy}$

(*a* multiplied by itself *x* times) multiplied by itself *y* times) = a multiplied by itself *x* ·*y* times

$$\underbrace{\overbrace{(a \cdot a \cdot \ldots)}^{\text{X times}} \cdot \overbrace{(a \cdot a \cdot \ldots)}^{\text{X times}} \cdot \ldots \overbrace{(a \cdot a \cdot \ldots)}^{\text{X times}}}_{\text{y times}}$$

7) $a^{x/y} = \sqrt[y]{a^x}$

$a^{1/y}$ is the number you need to multiply by itself y times to get a. $(a^{1/y})^y = a^{y/y} = a^1 = a$

So , $2^{1/2}$ is square root of 2, which is, $\sqrt{2}$ and $3^{1/3}$ is square root of 3, which is, $\sqrt[3]{3}$

Laws of indices

- 6) $(a^x)^y = a^{xy}$

$(10^3)^2 = 10^{(3x2)} = 10^6$
$1{,}000^2 = 1{,}000{,}000$

$(2^4)^2 = 2^{(2x4)} = 2^8$
$16^2 = 2^8 = 256$
$81 = (9)^2 = (3^2)^2 = 3^4 = 81$
$1/16 = (1/4)^2 = (2^{-2})^2 = 2^{-4} = 1/16$

Laws of indices

- 7) $a^{x/y} = \sqrt[y]{a^x}$

  $10^{(4/2)} = \sqrt[2]{10^4}$

  $10^2 = \sqrt[2]{10,000} = 100$

  $2^{(9/3)} = \sqrt[3]{2^9}$

  $2^3 = \sqrt[3]{512} = 8$

  $8 = 2^3 = 2^{6/2} = \sqrt[2]{64} = 8$

  $1/7 = (7)^{-1} = (7)^{-2/2} = \sqrt[2]{(1/49)} = 7$

Logarithmic identities

- 'Trivial'

Log form	Index form
$\log_b 1 = 0$	$b^0 = 1$
$\log_b b = 1$	$b^1 = b$

Logarithmic identities 2

- $y \cdot \log_b x = \log_b x^y$ $\qquad (b^x)^y = b^{xy}$

$$\boxed{x = b^{\log_b x}}$$
Definition of log

$$\boxed{\log_b b^z = z}$$
Definition of log

$$\log_b x^y = \log_b \left(b^{\log_b x}\right)^y = \log_b b^{y \times \log_b x} = y \times \log_b x$$

Logarithmic identities 2 examples

- $y \cdot \log_b x = \log_b x^y$ $(b^x)^y = b^{xy}$

Examples:

- $9 = 3 \cdot \log_2 8 = \log_2 8^3 = \log_2 512 = 9$

$$512 = (8)^3 = (2^3)^3 = 2^{3 \cdot 3} = 2^9 = 512$$

Logarithmic identities 3

Negative Identity

- $-\log_b x = \log_b (1/x)$ $\quad\quad$ $b^{-x} = 1/b^x$

Addition

- $\log_b x + \log_b y = \log_b xy$ $\quad\quad$ $b^x \cdot b^y = b^{(x + y)}$

Subtraction

- $\log_b x - \log_b y = \log_b x/y$ $\quad\quad$ $b^x / b^y = b^{(x - y)}$

Negative Identity

$b^{-y} = \frac{1}{b^y}$ (3rd law of indices)

$b^{\log_b x} = x$ (definition of log)

$$b^{-\log_b x} = \frac{1}{b^{\log_b x}} = \frac{1}{x}$$

Taking log from both sides of the equation

$$\log_b b^{-\log_b x} = \log_b \frac{1}{x}$$

$$-\log_b x = \log_b \frac{1}{x}$$

$\log_b b^y = y$ (definition of log)

Negative identity

Negative Identity

- $-\log_b x = \log_b (1/x)$ $\quad\quad$ $b^{-x} = 1/b^x$

Examples:

- $-3 = -\log_2 8 = \log_2 (1/8) = -3$ $\quad\quad$ $1/8 = 2^{-3} = 1/2^3 = 1/8$

Addition identity

$b^x \cdot b^y = b^{(x+y)}$ (4th law of indices)

$b^{\log_b x} = x$ (definition of log)

$$b^{\log_b x + \log_b y} = b^{\log_b x} \times b^{\log_b y} = xy$$

Taking log from both sides of the equation

$$\log_b b^{\log_b x + \log_b y} = \log_b xy$$

$$\boxed{\log_b x + \log_b y = \log_b xy}$$

$\log_b b^z = z$

Definition of log

Addition identity examples

Addition

- $\log_b x + \log_b y = \log_b xy$ $b^x \cdot b^y = b^{(x+y)}$

Examples:

- $5 = 2+3 = \log_2 4 + \log_2 8 = \log_2 4\cdot 8 = \log_2 32 = 5$

  $32 = 4 \cdot 8 = 2^2 \cdot 2^3 = 2^{(2+3)} = 2^5 = 32$

Subtraction Identity

$b^x \cdot b^y = b^{(x+y)}$ (4th law of indices)

$b^{\log_b x} = x$ (definition of log)

$b^{-\log_b x} = {}^1/_x$

(definition of log + 3rd law of indices)

$$b^{\log_b x - \log_b y} = b^{\log_b x} / b^{\log_b y} = {}^x/_y$$

Taking log from both sides of the equation

$$\log_b b^{\log_b x - \log_b y} = \log_b {}^x/_y$$

$\log_b b^z = z$

Definition of log

$$\log_b x - \log_b y = \log_b {}^x/_y$$

Subtraction identity examples

Subtraction

- $\log_b x - \log_b y = \log_b x/y \quad b^x / b^y = b^{(x - y)}$

Examples:

- $-1 = 2-3 = \log_2 4 - \log_2 8 = \log_2 4/8 = \log_2 1/2 = -1$

  $1/2 = 4 / 8 = 2^2 / 2^3 = 2^{(2 - 3)} = 2^{-1} = 1/2$
- $3 = 5-2 = \log_2 32 - \log_2 4 = \log_2 32/4 = \log_2 8 = 3$

  $8 = 32 / 4 = 2^5 / 2^2 = 2^{(5 - 2)} = 2^3 = 8$

Changing the base

- $\log_b x = \log_y x / \log_y b$

- leads to
  $\log_b x = 1/(\log_x b)$

Changing the base, examples 1

- $\log_b x = \log_y x / \log_y b$

Examples:

- $2 = \log_4 16 = \log_2 16 / \log_2 4 = 4/2 = 2$
- $4 = \log_3 81 = \log_5 81 / \log_5 3$

Changing the base, examples 2

- $\log_b x = 1/(\log_x b)$

Examples:

- $2 = \log_4 16 = 1/\log_{16} 4 = 1/(1/2) = 2$
- $4 = \log_3 81 = 1/ \log_{81} 3 = 1/(1/4) = 4$