Geometric Transformations

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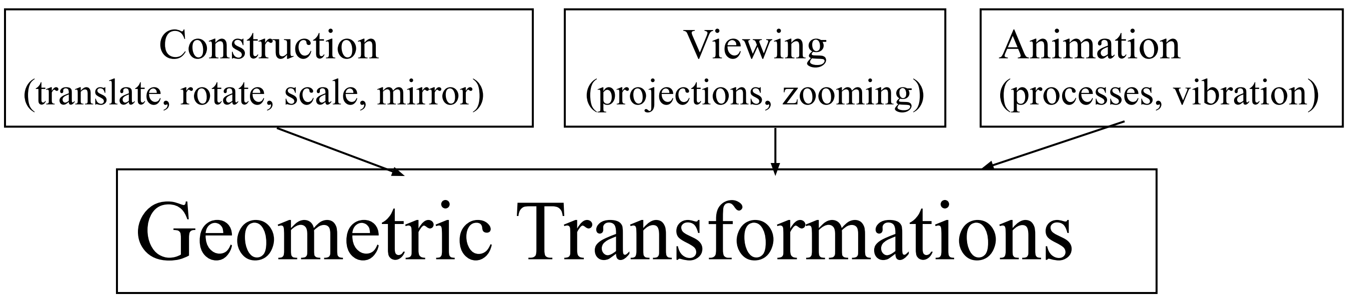
Intro & General Information



General Information

Transformation of a point is basic in GT. It can be formulated as follows:

Given a point P that belongs to a geometric model find the corresponding point P in the new position such that*

$$\mathbf{P}^* = f(\mathbf{P}, \text{transformation parameters})$$

- The transformation parameters should provide ONE-TO-ONE-MAPPING.
- Multiple transformations can be combined to yield a single transformation which should have the same effect as the sequential application of original ones. CONCATENATION /kənˌkatnˈāSH(ə)n/

Equation of $\mathbf{P}^*$ for graphics hardware should be in matrix notation:

$$\mathbf{P}^* = [T]\mathbf{P},$$

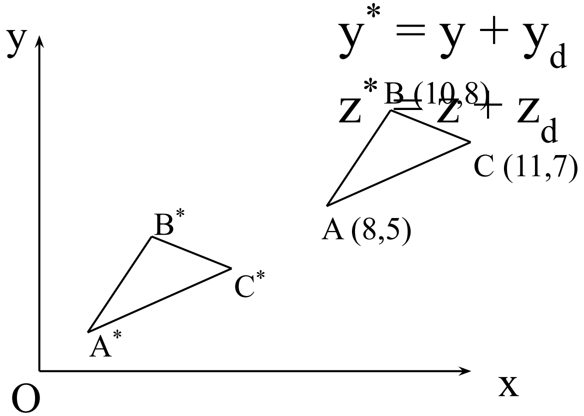
where [T] is the transformation matrix.

Translation

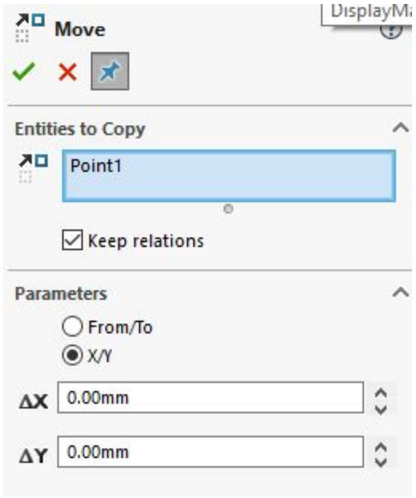
Translation is a rigid-body transformation (Euclidean) when each entity of the model remains parallel, or each point moves an equal distance in a given direction:

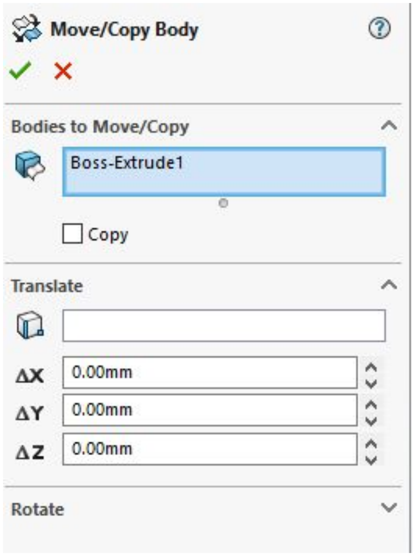
$\mathbf{P}^* = \mathbf{P} + \mathbf{d}$ (for both 2D and 3D). In a scalar form (for 3D):

$$x^* = x + x_d$$
$$y^* = y + y_d$$
$$z^* = z + z_d$$



Question: Find the coordinates of vertices A^*, B^*, and C^* of the translated triangle.
The distance vector of translation: $\mathbf{D} = [-7\ -4]^T$.
Verify that the lengths of the edges are unchanged.





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Scaling

Scaling is used to change the size of an entity or a model.

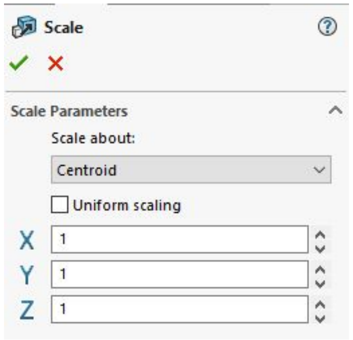
$$\mathbf{P}^* = [S]\mathbf{P}$$

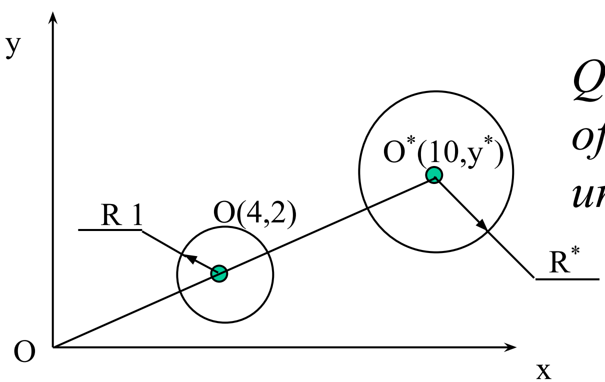
For general case $[S] = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$, where s_x, s_y, and s_z are the scaling factors in the X, Y, and Z directions respectively.

If $0 < s < 1$ - compression

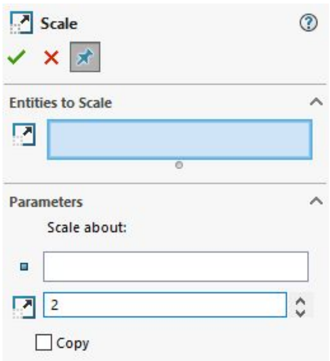
If $s > 1$ - stretching

$s_x = s_y = s_z$ - uniform scaling, otherwise - non-uniform





Question: The larger circle is the scaled copy of the smaller one. Can you say that we have a uniform scaling? Why? Define y and R*.*



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Mirror

Plane* => Negate the corresponding coordinate

Mirror through — Line* => Reflect through 2 planes intersecting at the axis

Point* => Reflect through 3 planes intersecting at the point

* plane - principal plane, line - X, Y, or Z axes, point - CS origin

$\mathbf{P}^{*} = [M]\mathbf{P}$,

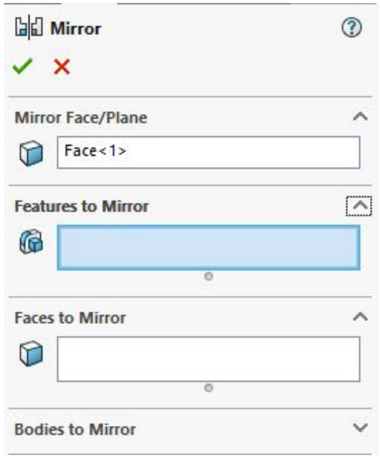
where $[M] = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix} \quad \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$

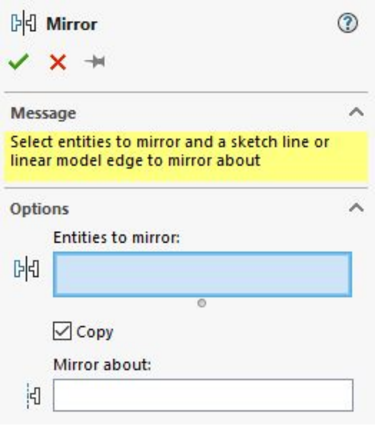
Question: *Define the signs (in the matrix) for the reflections (mirroring) through:*

a) x = 0, y = 0, z = 0 planes

b) X, Y, and Z axes

c) the CS origin



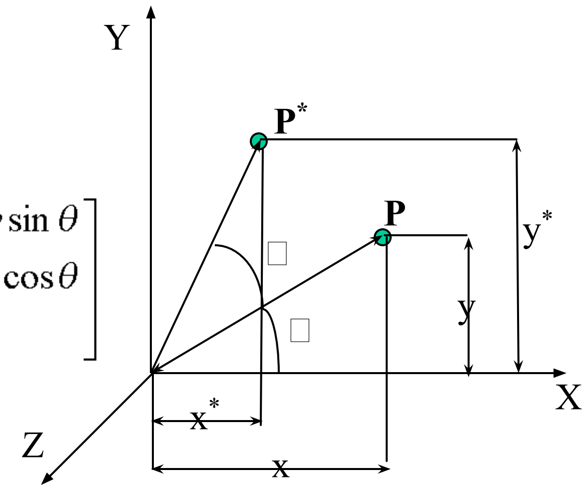


Rotation

Rotation is a non-commutative transformation (depends on sequence).

$$P^* = [R] \cdot P \qquad P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \cdot \cos\alpha \\ r \cdot \sin\alpha \\ z \end{bmatrix}$$

$$P^* = \begin{bmatrix} x^* \\ y^* \\ z^* \end{bmatrix} = \begin{bmatrix} r \cdot \cos(\alpha + \theta) \\ r \cdot \sin(\alpha + \theta) \\ z \end{bmatrix} = \begin{bmatrix} r \cdot \cos\alpha \cos\theta - r \cdot \sin\alpha \sin\theta \\ r \cdot \cos\alpha \sin\theta + r \cdot \sin\alpha \cos\theta \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$



$$[R_z] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$[R_y] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Question: *Let the length of a major and minor axes of an ellipse with the center on the origin of the CS be 2a and 2b respectively, and □ - the angle between the major axis and the x-axis. Then, derive the expression of an ellipse in the (O,x,y) system.*

Homogeneous Transformation - 1

When we scale then rotate, the transformed image is given by:

$$\mathbf{P}^* = ([R][S])\mathbf{P}$$

where [S], [R], [R] [S] are 3x3 transformation matrices. This is not the case for a translation ($\mathbf{P}^* = \mathbf{P} + \mathbf{d}$). The goal is to find a [D] such that

$$\mathbf{P} + \mathbf{d} = [D]\mathbf{P}$$

in order to perform valid matrix multiplication.

This is found by using a homogeneous coordinates.

Homogeneous Transformation maps n-dimensional space into (n+1)- dim.

3D representation of the point vector - $\mathbf{P} = [x, y, z]^T$

Homogeneous rep. of the same vector - $\mathbf{P} = [xw, yw, zw, w]^T$ where w = 1

Homogeneous Transformation - 2

The transformation matrices in new (homogeneous) representation:

$$[D]=\begin{bmatrix} 1 & 0 & 0 & x_d \\ 0 & 1 & 0 & y_d \\ 0 & 0 & 1 & z_d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [S]=\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

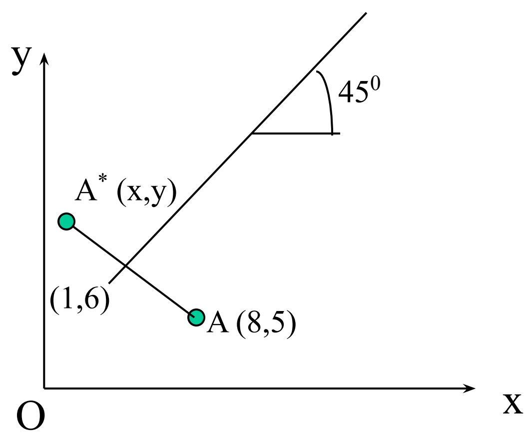
$$[M]=\begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\mathrm{R}]=\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition of Transformations

Now we are able to include all the transformations in a single matrix. In case of composition of transformations: $\mathbf{P}^* = [T_n][T_{n-1}]...[T_2][T_1]\mathbf{P}$, where $[T_i]$ are different transformation matrices.

<u>Sequence is important!</u>

Practice: Mirror point A through the given line and find x and y.



Another example

- Scale line AB about point M by factor of 2 and then mirror new line A’B’ about the origin.