NUFYP Mathematics

5.4 Differentiation 4. Sketching functions

Viktor Ten

Lecture Outline

- Sketching polynomials
  - Intercepts
  - End behavior
  - Derivatives
  - Conclusion & graph
- Sketching
- rational functions
  - Symmetries
  - Intercepts
  - Vertical asymptotes
  - Sign of *f(x)*
  - End behavior
  - Derivatives
  - Conclusion & graph
- Non-vertical asymptotes
  - Oblique (slant)
  - Curvilinear

Introduction

Historically, the term "curve sketching" meant using calculus to help draw the graph of a function by hand – the graph was the goal. Since graphs can now be produced with great precision using calculators and computers, the purpose of curve sketching has changed.

Today, we typically start with a graph produced by a calculator or computer, then use curve sketching to identify important features of the graph that the calculator or computer might have missed.

Thus, the goal of curve sketching is no longer the graph itself, but rather the information it reveals about the function.

Sketching polynomials

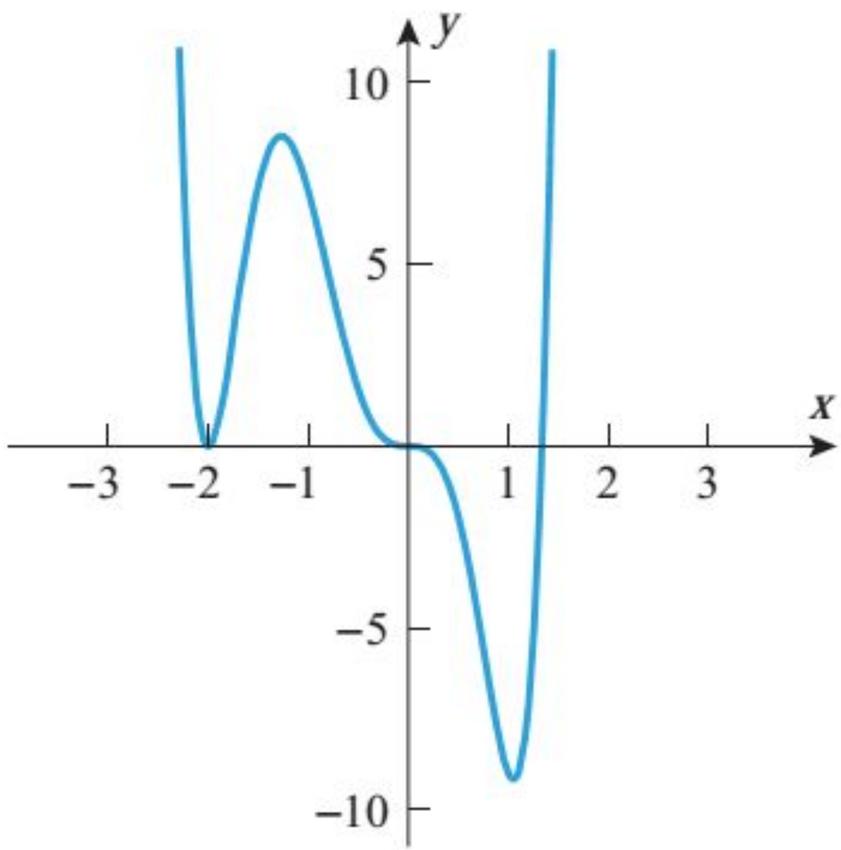
Define and describe (multiple or not) the roots of

$$y = x^3(3x-4)(x+2)^2$$

$x = 0$, multiplicity 3.

$x = \frac{4}{3}$, simple.

$x = -2$, multiplicity 2.



Sketching polynomials

A root of multiplicity 1 is called a ***simple root***.

A polynomial $p(x)$ has ***multiplicity m*** if $(x-r)^m$ divides $p(x)$ but $(x-r)^{m+1}$ does not.

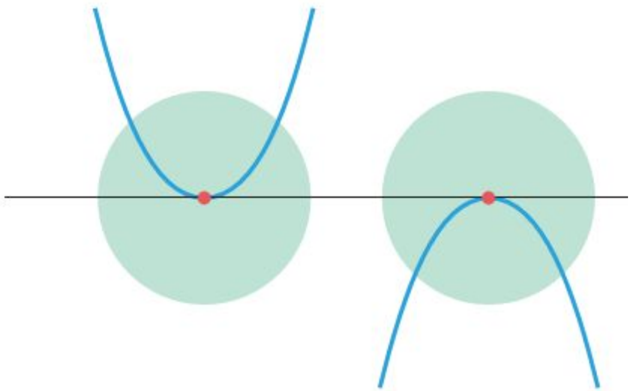
THE GEOMETRIC IMPLICATIONS OF MULTIPLICITY *Suppose that $p(x)$ is a polynomial with a root of multiplicity m at $x = r$.*

(a) *If m is even, then the graph of $y = p(x)$ is tangent to the x-axis at $x = r$, does not cross the x-axis there, and does not have an inflection point there.*

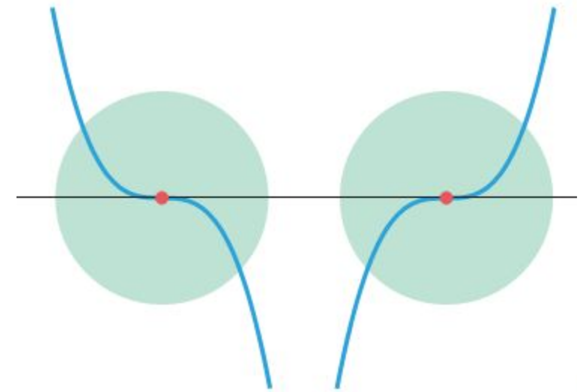
(b) *If m is odd and greater than 1, then the graph is tangent to the x-axis at $x = r$, crosses the x-axis there, and also has an inflection point there.*

(c) *If $m = 1$ (so that the root is simple), then the graph is not tangent to the x-axis at $x = r$, crosses the x-axis there, and may or may not have an inflection point there.*

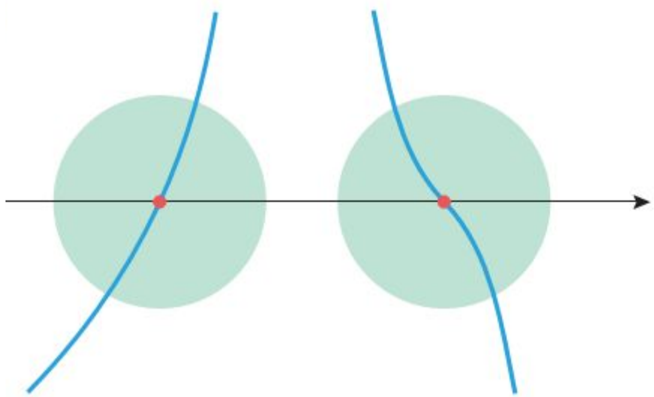
Sketching polynomials



Roots of even multiplicity



Roots of odd multiplicity (> 1)



Simple roots

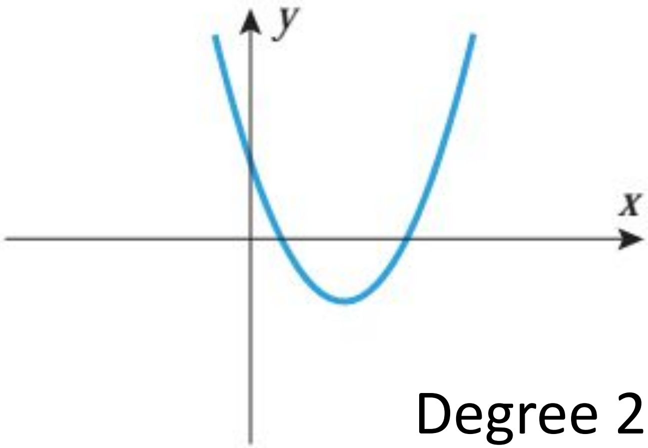
Sketching polynomials

Properties that are common to all polynomials:

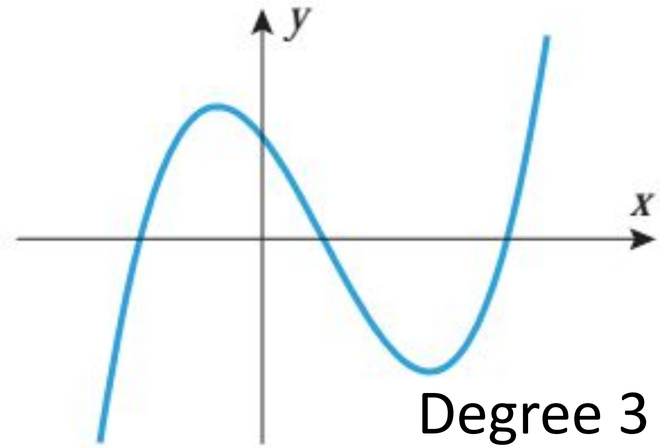
- The natural domain of a polynomial is $(+\infty, -\infty)$.
- Polynomials are continuous everywhere.
- Polynomials are differentiable everywhere, so their graphs have no corners or vertical tangent lines.
- The graph of a non-constant polynomial eventually increases or decreases without bound as $x \to +\infty$ and as $x \to -\infty$.
- The graph of a polynomial of degree n (≥ 2) has at most n x-intercepts, at most $n-1$ relative extrema, and $n-2$ inflection points.

Sketching polynomials

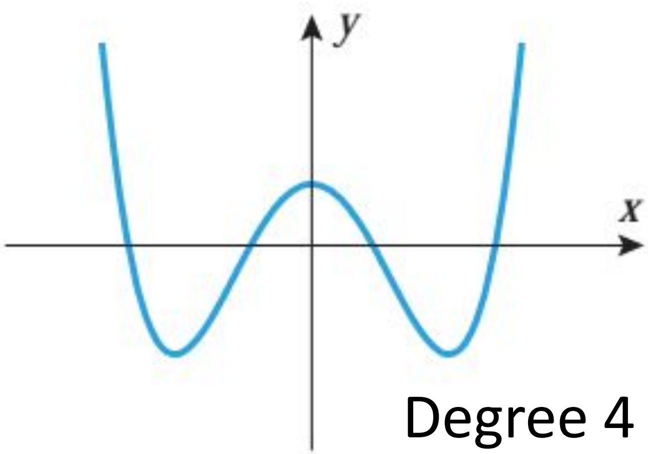
Degree?



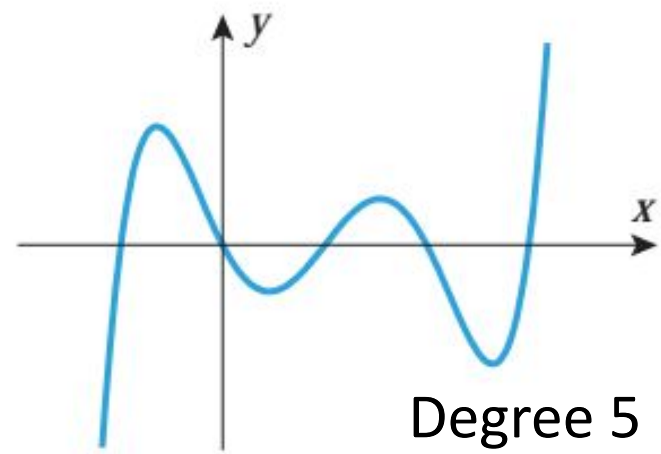
Degree 2



Degree 3



Degree 4



Degree 5

Sketching polynomials

Example 1. Sketch the graph of the equation $y = x^3 - 3x + 2$ and identify the locations of the intercepts, relative extrema, and inflection points.

Solution

- *x-intercepts:* Factoring the polynomial yields

$$x^3 - 3x + 2 = (x + 2)(x - 1)^2$$

which tells us that the x-intercepts are $x = -2$ and $x = 1$.

- *y-intercept:* Setting $x = 0$ yields $y = 2$.
- *End behavior:*

$$\lim_{x \to +\infty} (x^3 - 3x + 2) = \lim_{x \to +\infty} x^3 = +\infty$$

$$\lim_{x \to -\infty} (x^3 - 3x + 2) = \lim_{x \to -\infty} x^3 = -\infty$$

so the graph increases without bound as $x \to +\infty$ and decreases without bound as $x \to -\infty$.

Sketching polynomials

Example 1. Sketch the graph of the equation $y = x^3 - 3x + 2$ and identify the locations of the intercepts, relative extrema, and inflection points.

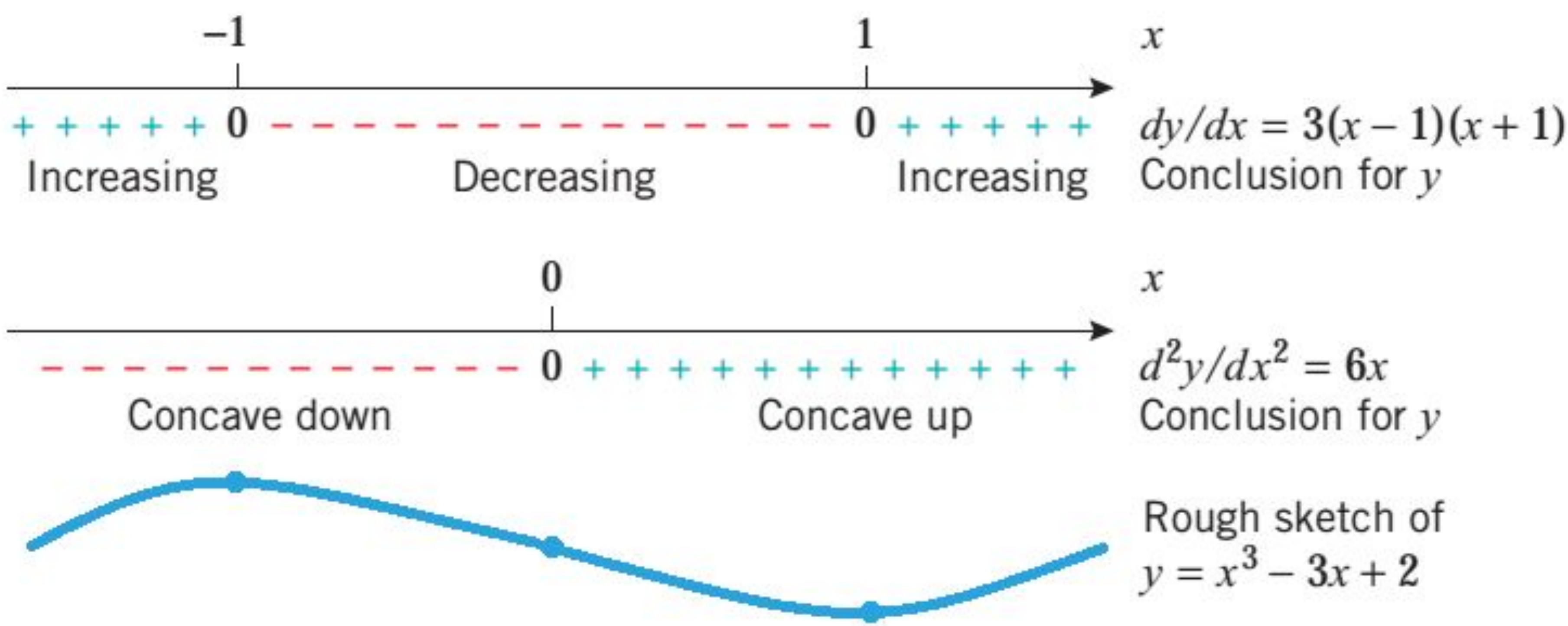
Solution

- *Derivatives:* $\dfrac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$ $\dfrac{d^2y}{dx^2} = 6x$
- *Increase, decrease, relative extrema, inflection points:*
  There are stationary points at $x = -1$ and $x = 1$. Since the sign of dy/dx changes from $+$ to $-$ at $x = -1$, there is a relative maximum there, and since it changes from $-$ to $+$ at $x = 1$, there is a relative minimum there.
  The sign of d^2y/dx^2 changes from $-$ to $+$ at $x = 0$, so there is an inflection point there.

Sketching polynomials

Example 1. Sketch the graph of the equation $y = x^3 - 3x + 2$ and identify the locations of the intercepts, relative extrema, and inflection points.

Solution

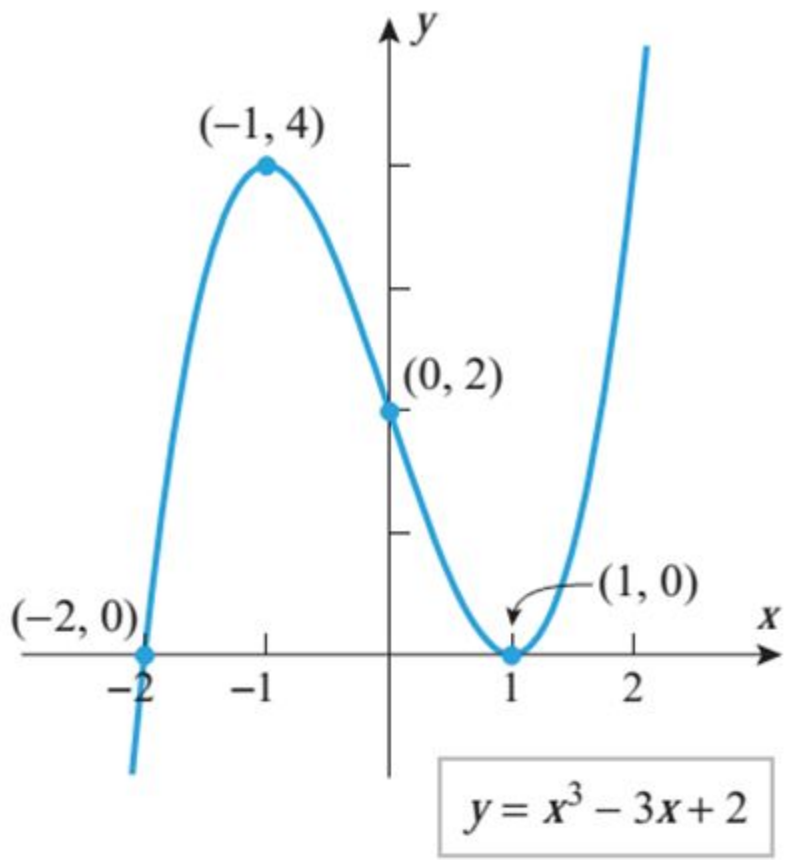


Sketching polynomials

Example 1. Sketch the graph of the equation $y = x^3 - 3x + 2$ and identify the locations of the intercepts, relative extrema, and inflection points.

Solution

- *Final sketch:*



Properties of graphs

In many problems, the properties of interest in the graph of a function are:

- symmetries
- x-intercepts
- relative extrema
- intervals of increase and decrease
- asymptotes
- periodicity
- y-intercepts
- concavity
- inflection points
- behavior as $x \to +\infty$ or as $x \to -\infty$

Some of these properties may not be relevant in certain cases; for example, asymptotes are characteristic of rational functions but not of polynomials, and periodicity is characteristic of trigonometric functions but not of polynomial or rational functions.

Thus, when analyzing the graph of a function f, it helps to know something about the general properties of the family to which it belongs.

Sketching rational functions

Graphing a rational function $f(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are ***polynomials*** and have ***no common factors***.

Step 1. **(symmetries).**
Step 2. **(x- and y-intercepts).**
Step 3. **(vertical asymptotes).**
Step 4. **(sign of $f(x)$).**
Step 5. **(end behavior).**
Step 6. **(derivatives).**
Step 7. **(conclusions and graph).**

Sketching rational functions

Graphing a rational function $f(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are ***polynomials*** and have ***no common factors***.

Step 1.
(symmetries). Determine whether there is symmetry about the y-axis or the origin.

Step 2.
(x- and y-intercepts). Find the $x-$ and $y-$intercepts.

Step 3.
(vertical asymptotes). Find the values of x for which $Q(x) = 0$. The graph has a vertical asymptote at each such value.

Sketching rational functions

Graphing a rational function $f(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are ***polynomials*** and have ***no common factors***.

Step 4.

(sign of $f(x)$). The only places where $f(x)$ can change sign are at the x-intercepts or vertical asymptotes. Mark the points on the x-axis at which these occur and calculate a sample value of $f(x)$ in each of the open intervals determined by these points. This will tell you whether $f(x)$ is positive or negative over that interval.

Sketching rational functions

Graphing a rational function $f(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are ***polynomials*** and have ***no common factors***.

Step 5.
(end behavior). Determine the end behavior of the graph by computing the limits of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. If either limit has a finite value L, then the line $y = L$ is a horizontal asymptote.

Step 6.
(derivatives). Find $f'(x)$ $f''(x)$.

Sketching rational functions

Graphing a rational function $f(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are ***polynomials*** and have ***no common factors***.

Step 7.
(conclusion and graph). Analyze the sign changes of $f'(x)$ and $f''(x)$ to determine the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down. Determine the locations of all stationary points, relative extrema, and inflection points. Use the sign analysis of $f(x)$ to determine the behavior of the graph in the vicinity of the vertical asymptotes. Sketch a graph of $f(x)$ that exhibits these conclusions.

Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution

- *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.
- x- and y-*intercepts:* Setting $y = 0$ yields the x-intercepts $x = -2$ and $x = 2$. Setting $x = 0$ yields the y-intercept $y = \frac{1}{2}$.
- *Vertical asymptotes:* We observed above that the numerator and denominator of y have no common factors, so the graph has vertical asymptotes at the points where the denominator of y is zero, namely, at $x = -4$ and $x = 4$.

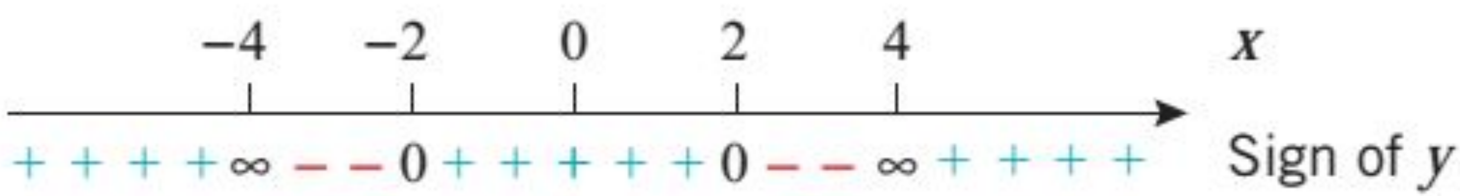
Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution

- *Sign of y:* The set of points where x-intercepts or vertical asymptotes occur is $\{-4, -2, 2, 4\}$. These points divide the x-axis into the open intervals

$$(-\infty, -4), \quad (-4, -2), \quad (-2, 2), \quad (2, 4), \quad (4, +\infty)$$



Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution

- *End behavior:* The limits

$$\lim_{x \to +\infty} \frac{2x^2 - 8}{x^2 - 16} = \lim_{x \to +\infty} \frac{2 - (8/x^2)}{1 - (16/x^2)} = 2$$

$$\lim_{x \to -\infty} \frac{2x^2 - 8}{x^2 - 16} = \lim_{x \to -\infty} \frac{2 - (8/x^2)}{1 - (16/x^2)} = 2$$

  yield the horizontal asymptote $y = 2$.

- *Derivatives:* $$\frac{dy}{dx} = \frac{(x^2 - 16)(4x) - (2x^2 - 8)(2x)}{(x^2 - 16)^2} = -\frac{48x}{(x^2 - 16)^2}$$

$$\frac{d^2y}{dx^2} = \frac{48(16 + 3x^2)}{(x^2 - 16)^3}$$

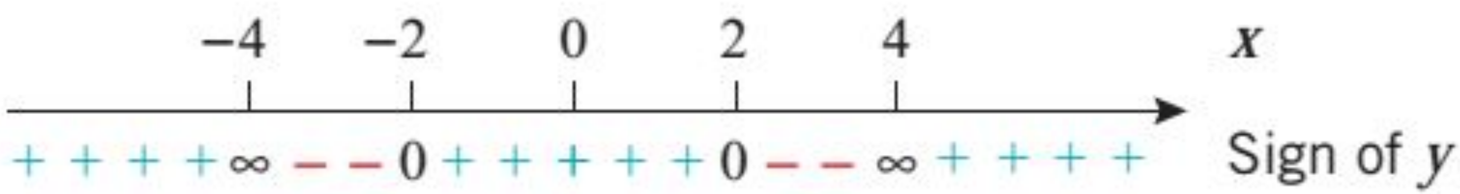
Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution

- *Conclusions and graph:*

The sign analysis of y reveals the behavior of the graph in the vicinity of the vertical asymptotes:
The graph increases without bound as $x \to -4^-$
and decreases without bound as $x \to -4^+$;
and the graph decreases without bound as $x \to 4^-$
and increases without bound as $x \to 4^+$.

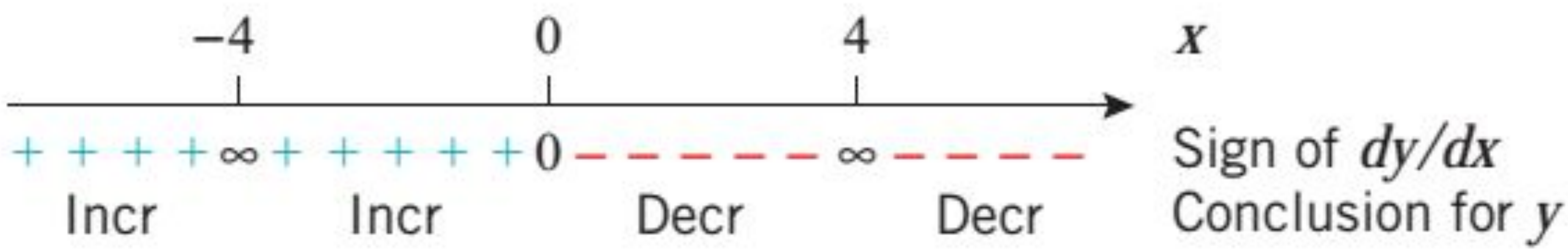


Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution

- *Conclusions and graph:* $\dfrac{dy}{dx} = -\dfrac{48x}{(x^2 - 16)^2}$



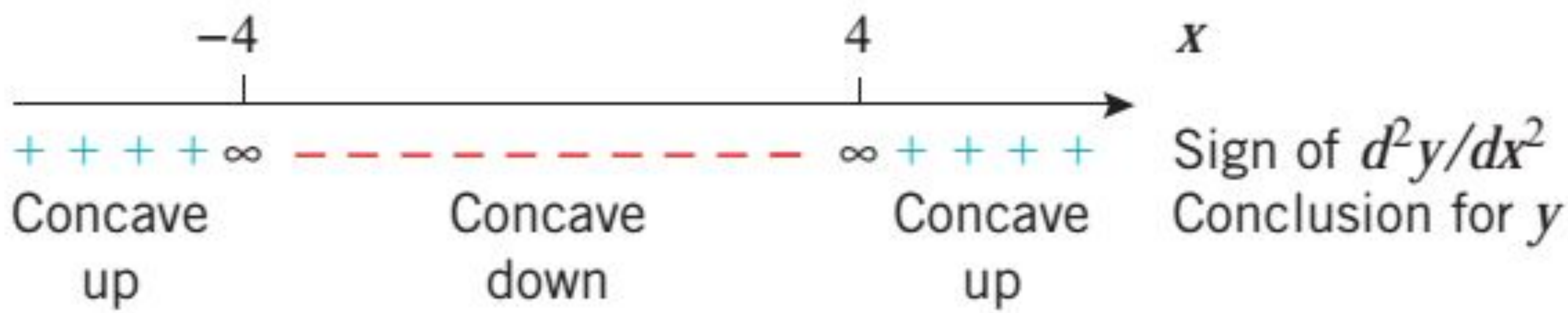
The sign analysis of dy/dx shows that the graph is increasing to the left of $x = 0$ and is decreasing to the right of $x = 0$.
Thus, there is a relative maximum at the stationary point $x = 0$. There are no relative minima.

Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution

- *Conclusions and graph:* $\dfrac{d^2y}{dx^2} = \dfrac{48(16 + 3x^2)}{(x^2 - 16)^3}$

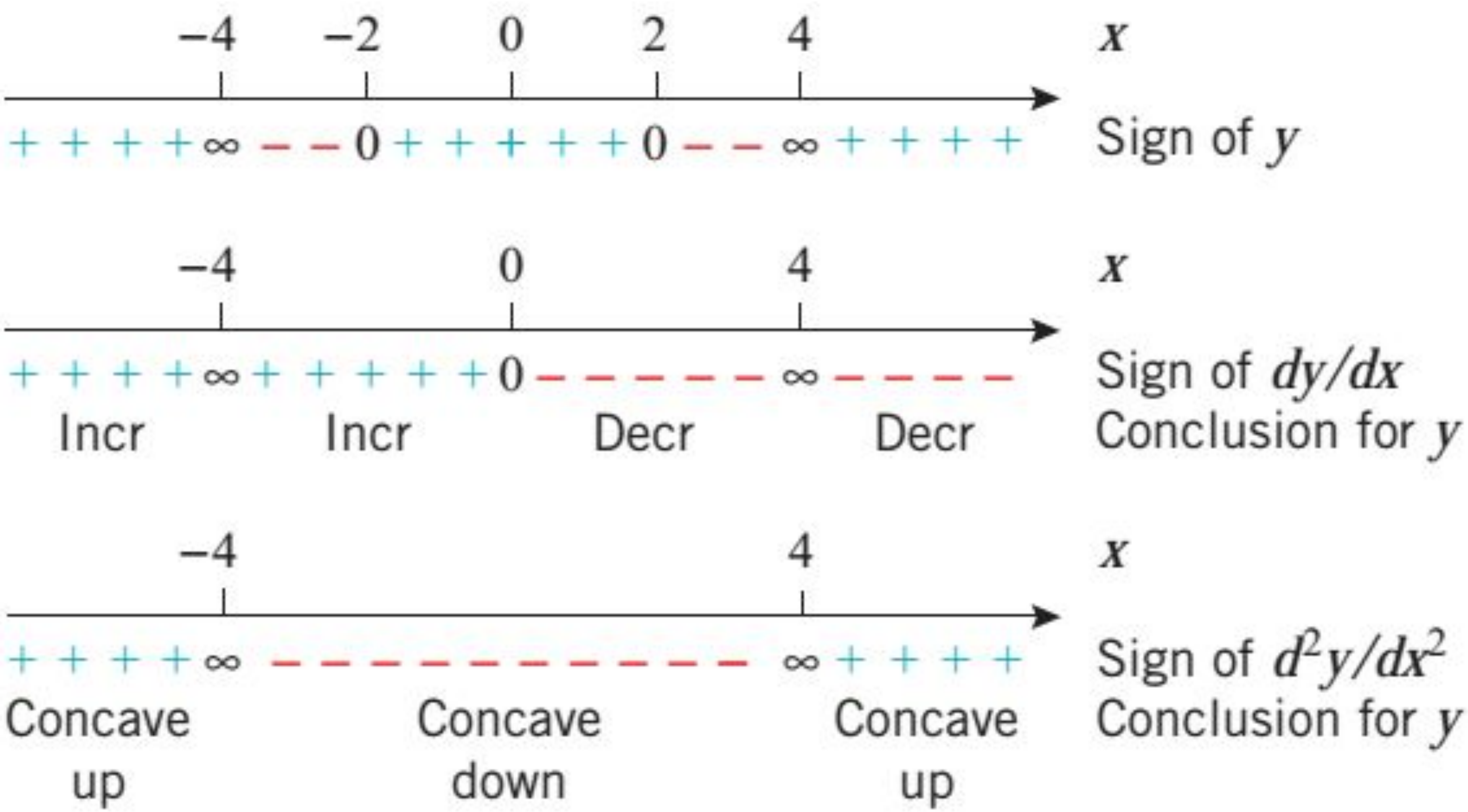


The sign analysis of d^2y/dx^2 shows that the graph
is concave up to the left of $x = -4$,
is concave down between $x = -4$ and $x = 4$, and
is concave up to the right of $x = 4$.
There are no inflection points.

Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution • *Conclusions and graph:*

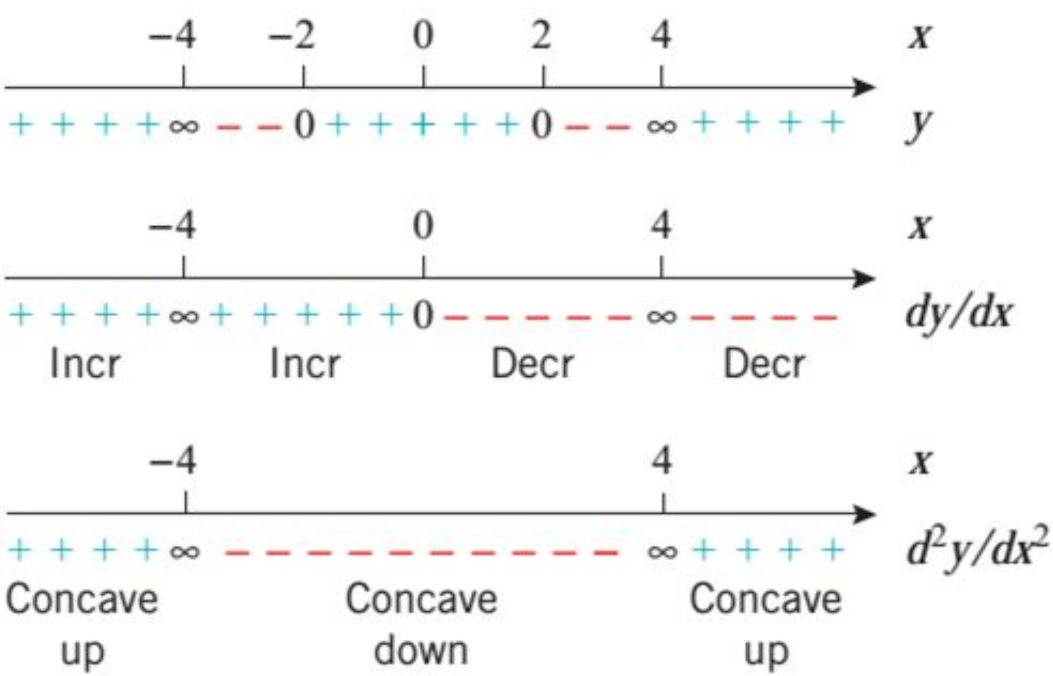


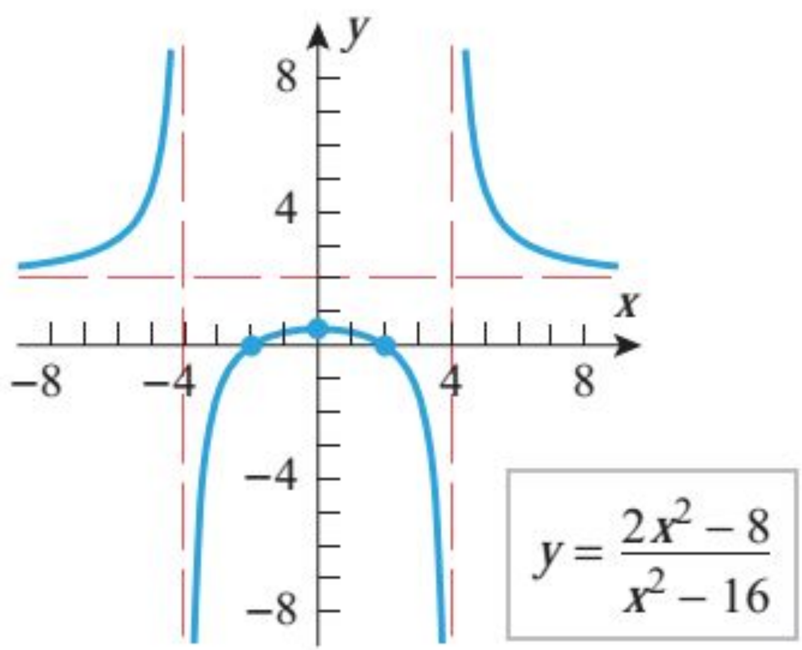
Sketching rational functions

Example 2. Sketch a graph of the equation $y = \dfrac{2x^2 - 8}{x^2 - 16}$

Solution

- *Conclusions and graph:*





Sketching rational functions

Example 3. Sketch a graph of $y = \dfrac{x^2 - 1}{x^3}$

Solution

- *Symmetries:* Replacing x by $-x$ and y by $-y$ yields an equation that simplifies to the original equation, so the graph is symmetric about the origin.
- *x- and y-intercepts:* Setting $y = 0$ yields the x-intercepts $x = -1$ and $x = 1$. Setting $x = 0$ leads to a division by zero, so there is no y-intercept.
- *Vertical asymptotes:* Setting $x^3 = 0$ yields the solution $x = 0$. This is not a root of $x^2 - 1$, so $x = 0$ is a vertical asymptote.

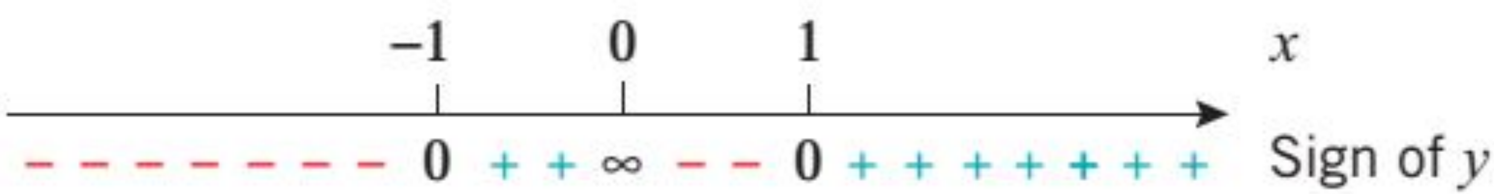
Sketching rational functions

Example 3. Sketch a graph of $y = \dfrac{x^2 - 1}{x^3}$

Solution

- *Sign of y:* The set of points where x-intercepts or vertical asymptotes occur is $\{-1, 0, 1\}$. These points divide the x-axis into the open intervals

$$(-\infty, -1), \quad (-1, 0), \quad (0, 1), \quad (1, +\infty)$$



Sketching rational functions

Example 3. Sketch a graph of $y = \dfrac{x^2-1}{x^3}$

Solution

- *End behavior:* The limits $\displaystyle\lim_{x\to+\infty} \frac{x^2-1}{x^3} = \lim_{x\to+\infty}\left(\frac{1}{x} - \frac{1}{x^3}\right) = 0$

$$\lim_{x\to-\infty} \frac{x^2-1}{x^3} = \lim_{x\to-\infty}\left(\frac{1}{x} - \frac{1}{x^3}\right) = 0$$

yield the horizontal asymptote $y = 0$.

- *Derivatives:*

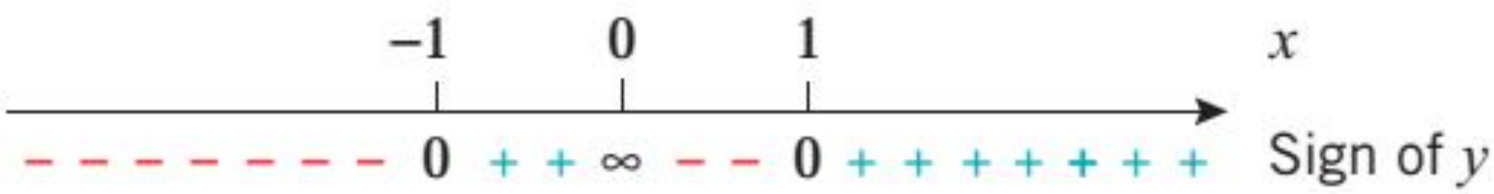
$$\frac{dy}{dx} = \frac{x^3(2x) - (x^2-1)(3x^2)}{(x^3)^2} = \frac{3-x^2}{x^4} = \frac{(\sqrt{3}+x)(\sqrt{3}-x)}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{x^4(-2x) - (3-x^2)(4x^3)}{(x^4)^2} = \frac{2(x^2-6)}{x^5} = \frac{2(x-\sqrt{6})(x+\sqrt{6})}{x^5}$$

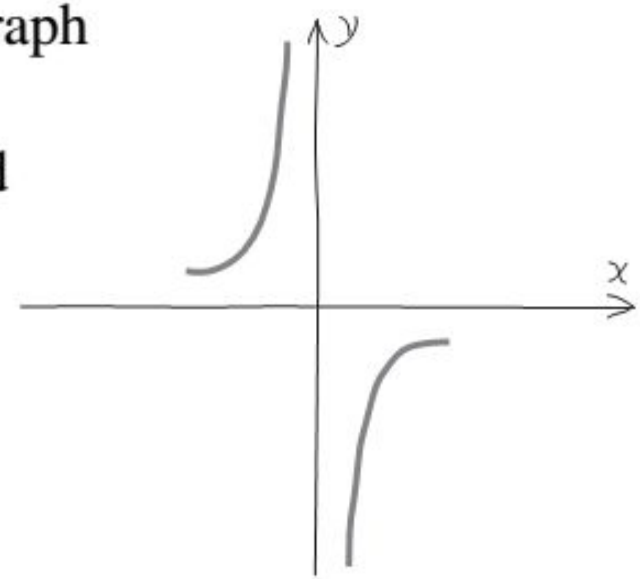
Sketching rational functions

Example 3. Sketch a graph of $y = \dfrac{x^2 - 1}{x^3}$

Solution



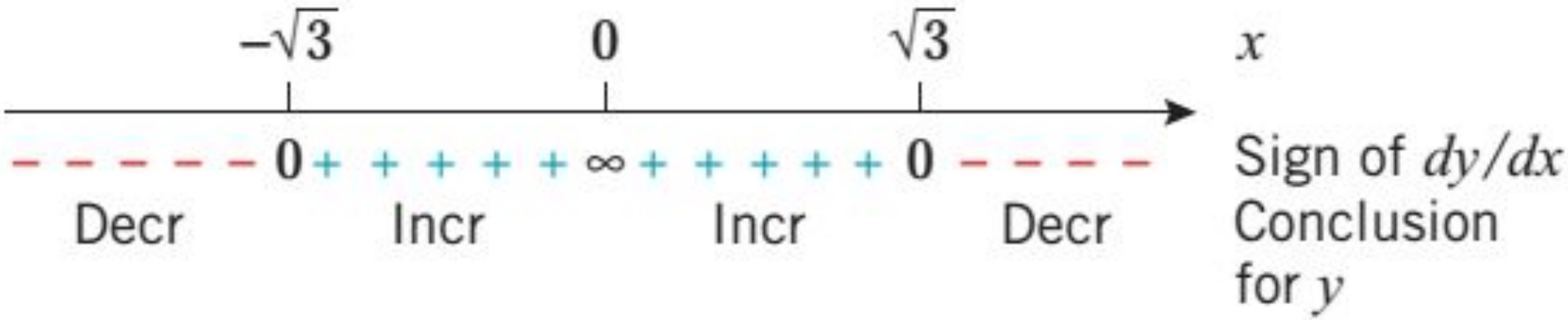
The sign analysis of y reveals the behavior of the graph in the vicinity of the vertical asymptote $x = 0$: The graph increases without bound as $x \to 0^-$ and decreases without bound as $x \to 0^+$.



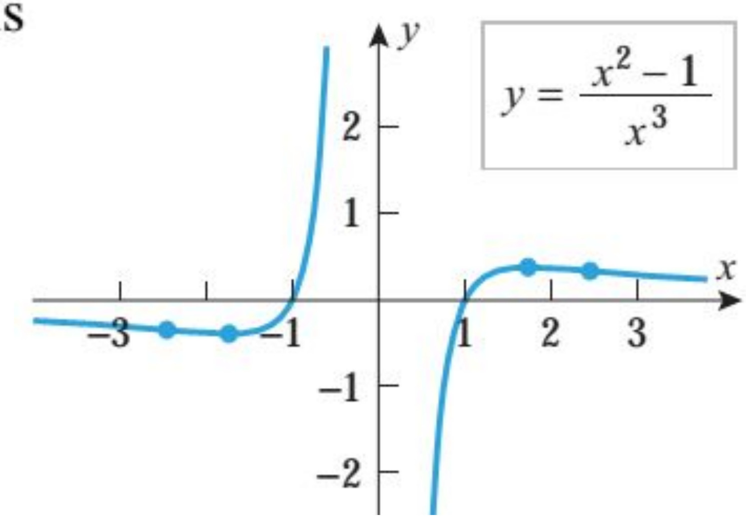
Sketching rational functions

Example 3. Sketch a graph of $y = \dfrac{x^2 - 1}{x^3}$

Solution



The sign analysis of dy/dx shows that there is a relative minimum at $x = -\sqrt{3}$ and a relative maximum at $x = \sqrt{3}$.



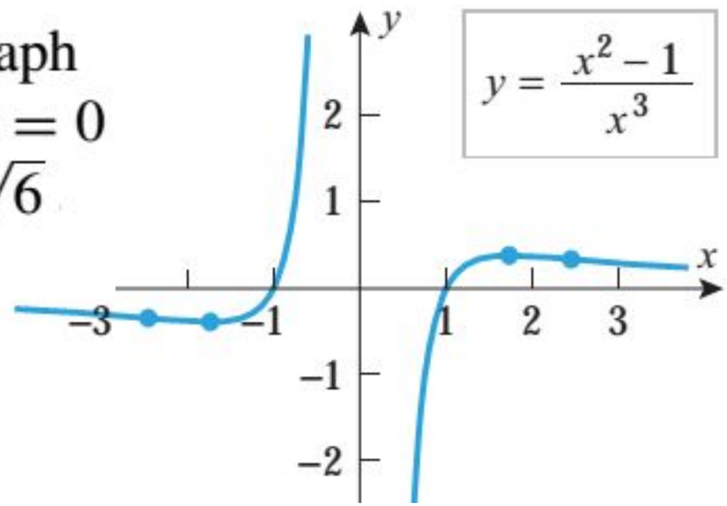
Sketching rational functions

Example 3. Sketch a graph of $y = \dfrac{x^2 - 1}{x^3}$

Solution

	$-\sqrt{6}$		0		$\sqrt{6}$		x
$- - -$	0	$+ + + + + +$	∞	$- - - - - -$	0	$+ + +$	Sign of d^2y/dx^2
Concave down		Concave up		Concave down		Concave up	Conclusion for y

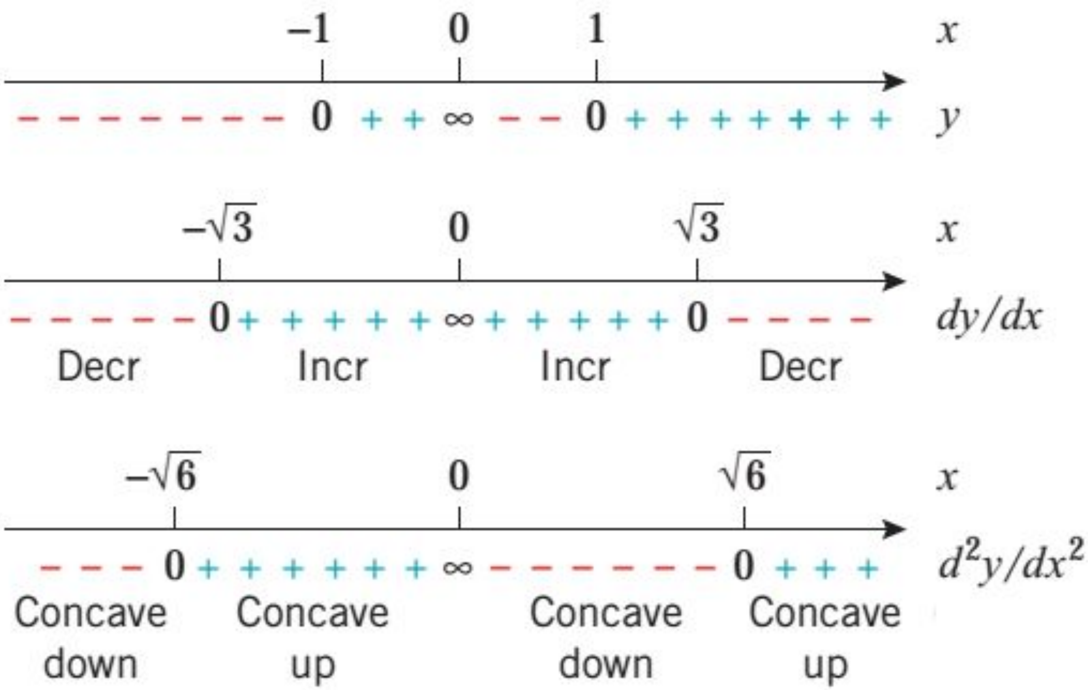
The sign analysis of d^2y/dx^2 shows that the graph changes concavity at the vertical asymptote $x = 0$ and that there are inflection points at $x = -\sqrt{6}$ and $x = \sqrt{6}$.

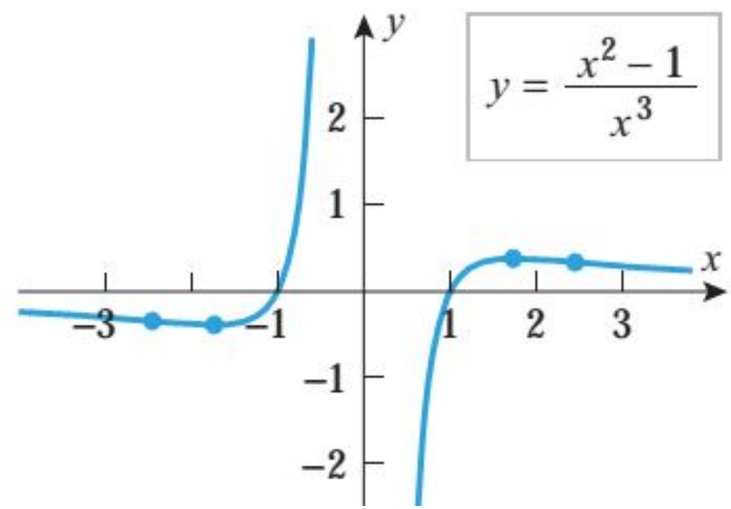


Sketching rational functions

Example 3. Sketch a graph of $y = \dfrac{x^2 - 1}{x^3}$

Solution





Oblique (slant) and curvilinear asymptotes

If the numerator of a rational function has greater degree than the denominator, then other kinds of "asymptotes" are possible.

In general, if $f(x) = P(x)/Q(x)$ is a rational function, then we can find quotient and remainder polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x) + \frac{r(x)}{Q(x)}$$

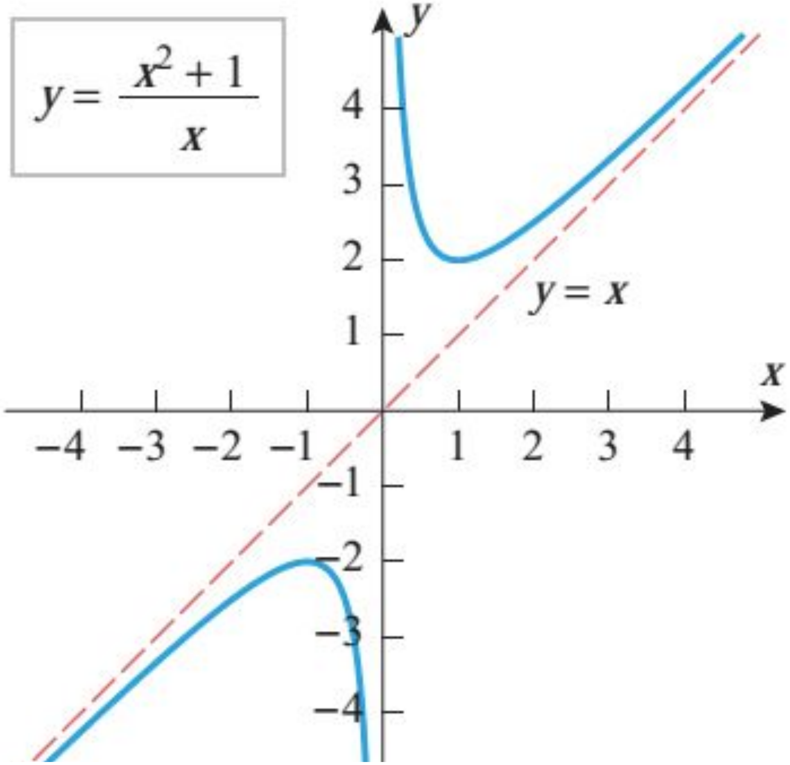
and the degree of $r(x)$ is less than the degree of $Q(x)$.

Then $r(x)/Q(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$, so $y = q(x)$ is an asymptote of f.

This asymptote will be an oblique line if the degree of $P(x)$ is one greater than the degree of $Q(x)$, and it will be curvilinear if the degree of $P(x)$ exceeds that of $Q(x)$ by two or more.

Slant (oblique) asymptotes

Example 4. Consider a rational function $f(x) = \dfrac{x^2+1}{x}$



By division we can rewrite this as

$$f(x) = x + \frac{1}{x}$$

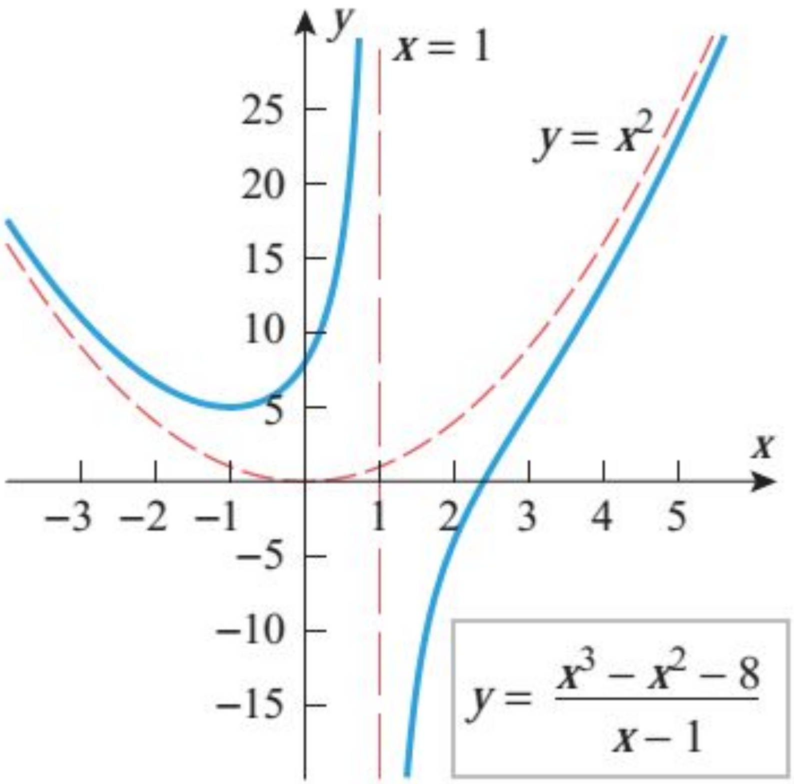
Since the second term approaches 0 as $x \to +\infty$ or as $x \to -\infty$, it follows that

$$\boxed{(f(x) - x) \to 0 \quad \text{as } x \to +\infty \text{ or as } x \to -\infty}$$

Geometrically, this means that the graph of $y = f(x)$ eventually gets closer and closer to the line $y = x$ as $x \to +\infty$ or as $x \to -\infty$. The line $y = x$ is called an ***oblique*** or ***slant asymptote*** of f.

Curvilinear asymptotes

Example 5. Consider a rational function $g(x) = \dfrac{x^3 - x^2 - 8}{x - 1}$



By division we can rewrite this as

$$g(x) = x^2 - \frac{8}{x - 1}$$

Since the second term approaches 0 as $x \to +\infty$ or as $x \to -\infty$, it follows that

$$\boxed{(g(x) - x^2) \to 0 \quad \text{as } x \to +\infty \text{ or as } x \to -\infty}$$

Geometrically, this means that the graph of $y = g(x)$ eventually gets closer and closer to the parabola $y = x^2$ as $x \to +\infty$ or as $x \to -\infty$. The parabola $y = x^2$ is called a ***curvilinear asymptote*** of g.

Learning outcomes

5.4.1. Sketch a graph of a polynomial function.

5.4.2. Sketch a graph of a rational function.

5.4.3. Define an oblique (slant) asymptote of a rational function, if exists.

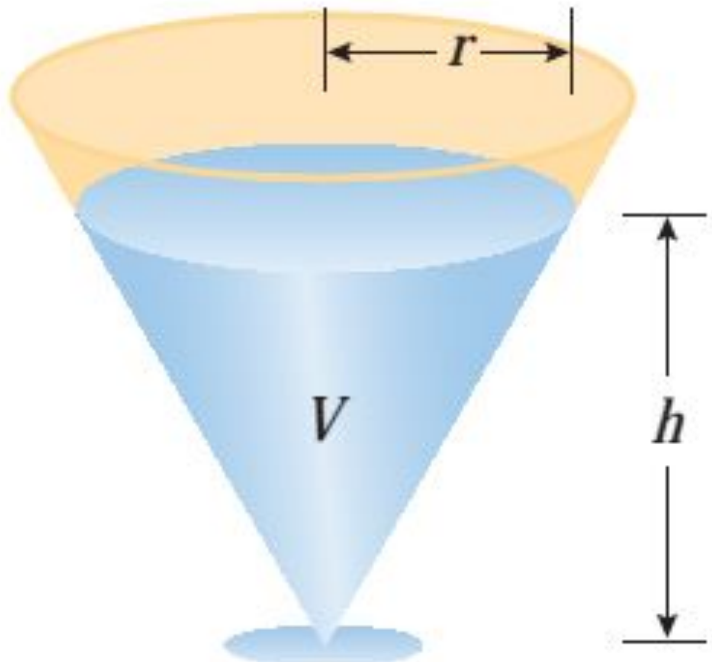
5.4.4. Define a curvilinear asymptote of a rational function, if exists.

Formulae

Graphing a rational function $f(x) = P(x)/Q(x)$ if $P(x)$ and $Q(x)$ are ***polynomials*** and have ***no common factors***.

Step 1. **(symmetries).**
Step 2. **(x- and y-intercepts).**
Step 3. **(vertical asymptotes).**
Step 4. **(sign of $f(x)$).**
Step 5. **(end behavior).**
Step 6. **(derivatives).**
Step 7. **(conclusions and graph).**

Preview activity: Differentiation 5



Suppose a liquid draining process. As the liquid drains, its volume V, height h, an radius r are functions of the elapsed time t, and at each instant these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h$$

Find the rate of change of the volume V with respect to the time t.