Mechanics of Material
Chapter II

Stress and Strain – Axial Loading

Stress and Strain
Contents

Stress and Strain
Axial loading

- Suitability of a structure or machine may depend on the deformations in the structure as well as the stresses induced under loading. Statics analyses alone are not sufficient.

- Considering structures as deformable allows determination of member forces and reactions which are statically indeterminate.

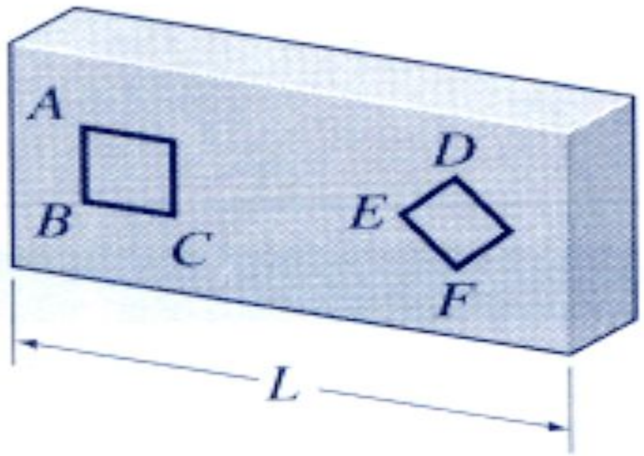
- Determination of the stress distribution within a member also requires consideration of deformations in the member.

Displacement

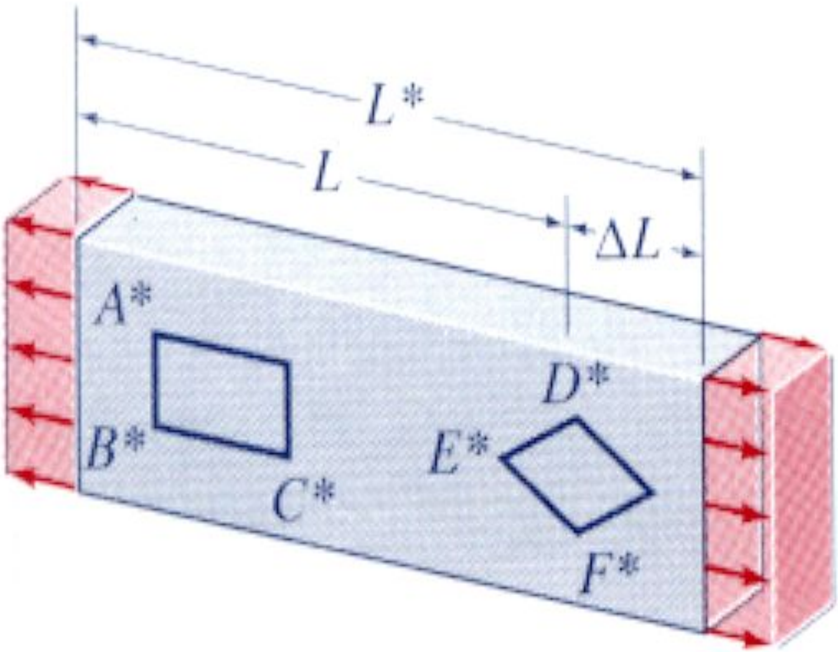
Movement of a point w.r.t. a reference system. Maybe caused by translation and or rotation of object (rigid body). Change in shape or size related to displacements are called deformations. Change in linear dimension causes deformation δ

Deformation

Includes changes in both lengths and angles.



(*a*) The undeformed bar.



(*b*) The deformed bar.

Strain

A quantity used to measure the intensity of deformation.
Stress is used to measure the intensity of internal force.

Normal strain, ε, used to measure change in size.
Shear strain, γ, used to measure change in shape.

Axial Strain at a Point

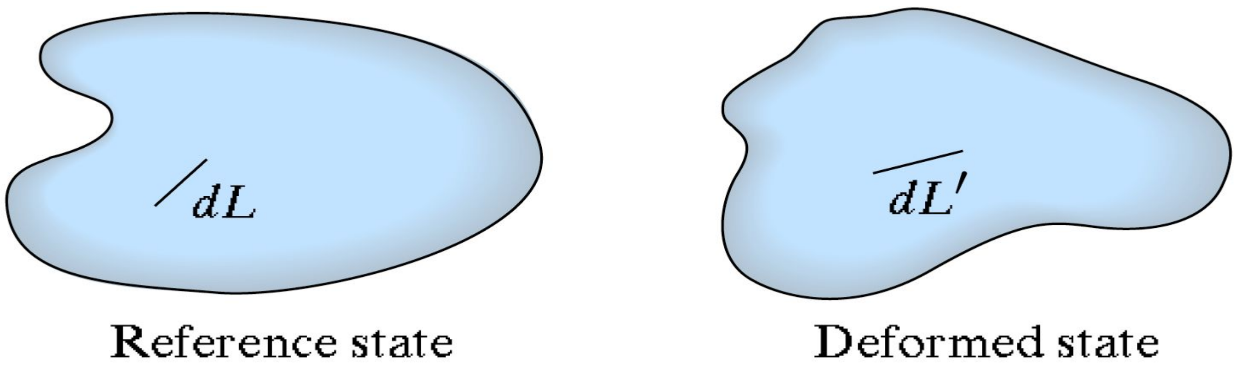


FIGURE 6-1 Material line element in the reference and deformed states.

Bedford/Liechti, Mechanics of Materials, 1e, ©2001, Prentice Hall

$$\varepsilon = \frac{dL' - dL}{dL}$$

$$= \lim_{\Delta L \to 0} \left(\frac{\Delta \delta_n}{\Delta L} \right)$$

Axial Strain at a Point

$$\varepsilon = \frac{dL' - dL}{dL}$$

If the bar stretches ($dL' > dL$), the strain is positive and called a tensile strain.

If the bar contracts ($dL' < dL$), the strain is negative and called a compressive strain.

Normal Strain/ **Axial Strain at a Point**

$$\varepsilon = \frac{dL' - dL}{dL} \rightarrow \quad \varepsilon dL = dL' - dL$$

$$(1+\varepsilon)dL = dL'$$

$$L' = \int_L (1+\varepsilon)dL = L + \int_L \varepsilon\, dL$$

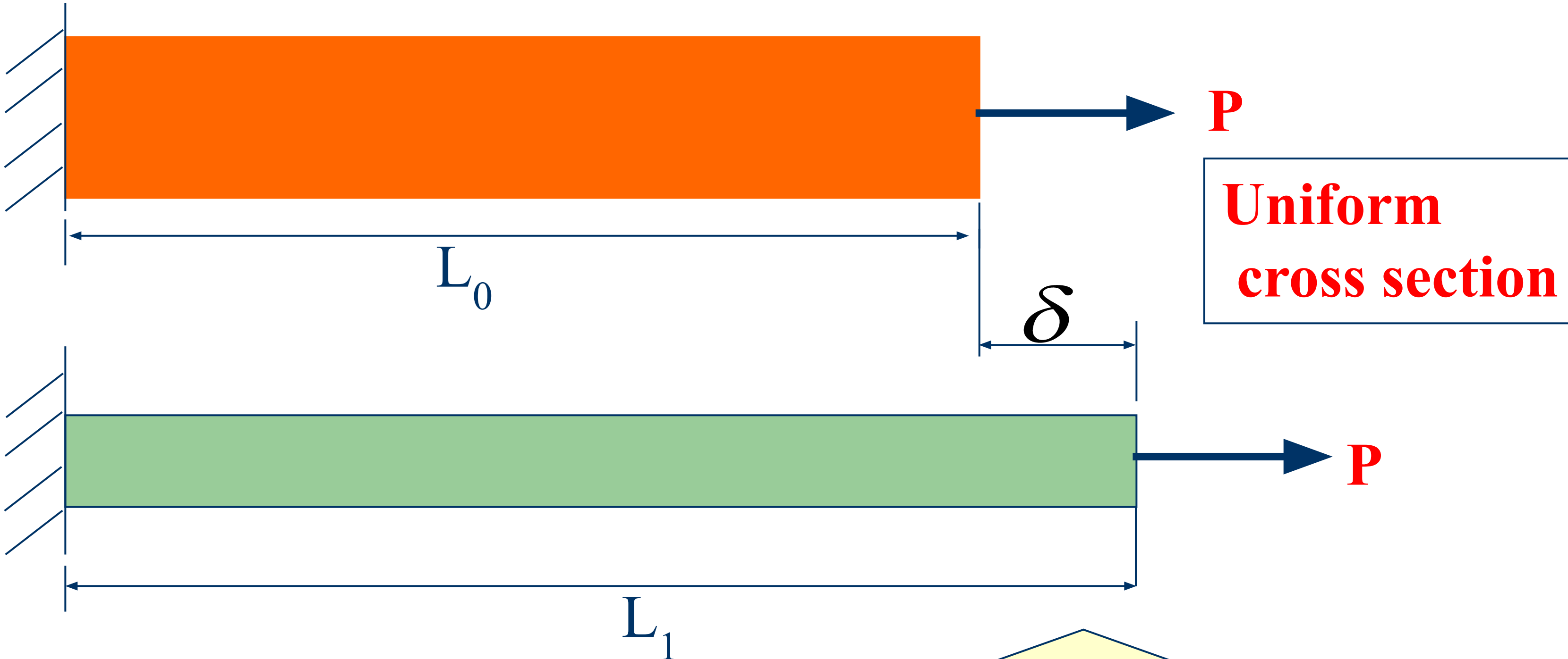
$$\delta = L' - L = \int_L \varepsilon\, dL$$

$$\delta = L' - L = \varepsilon L$$

$$\varepsilon = \frac{L' - L}{L} = \frac{\delta}{L}$$

Normal Strain

Normal Strain: is the deformation of the Member per unit length.



Normal strain is essentially:

<u>Change in Length</u>
Original Length

$$\varepsilon = \frac{\delta}{L_0} = \text{normal strain}$$

(Dimensionless)

$$with \qquad \delta = L_1 - L_0$$

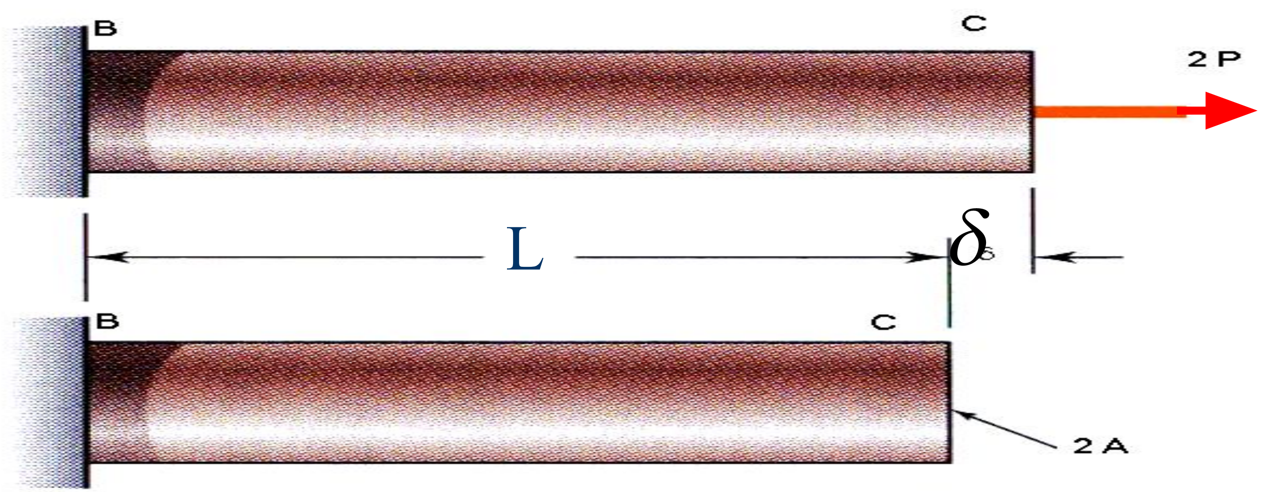
2.1 Normal Strain

$$\varepsilon_{avg} = \frac{\Delta L}{L} = \frac{L' - L}{L}$$

If the bar stretches ($L' > L$), the strain is positive and called a tensile strain.

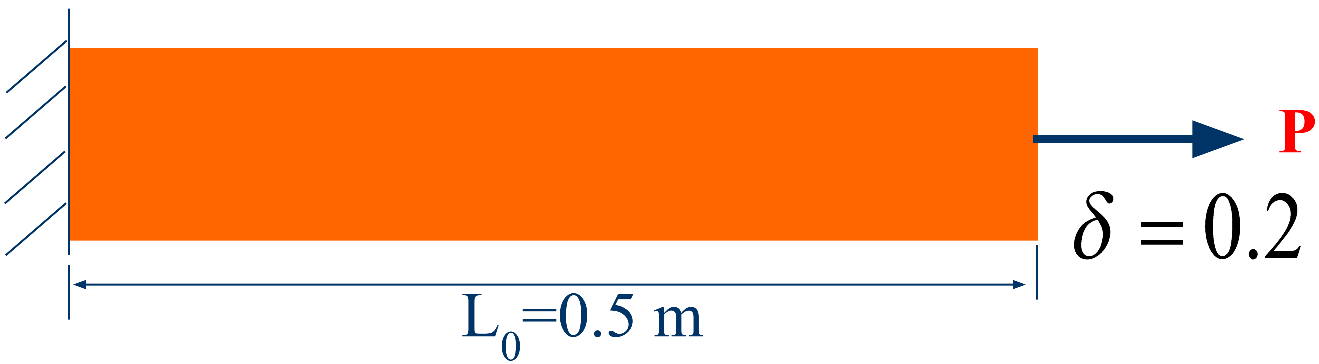
If the bar contracts ($L' < L$), the strain is negative and called a compressive strain.

2.1 Normal Strain
Examples



$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$

$$\varepsilon = \frac{\delta}{L}$$



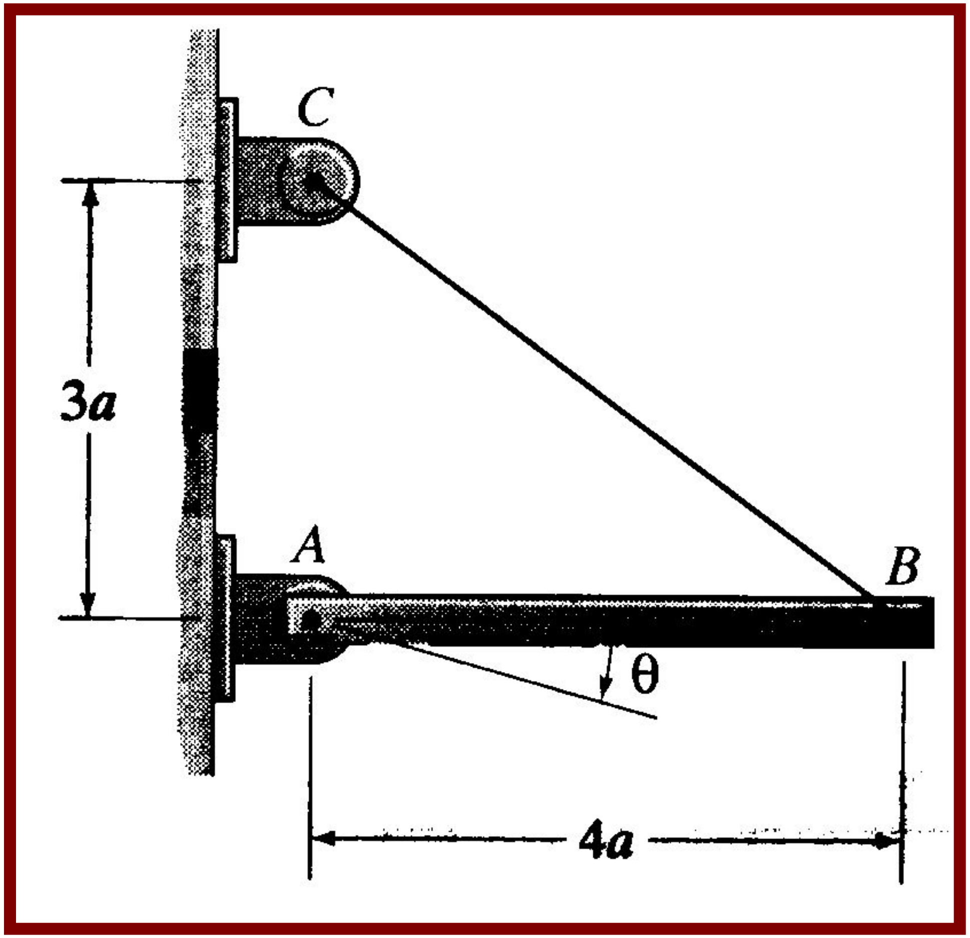
$$\delta = 0.2\ mm$$

$$\varepsilon = \frac{\delta}{L_0} = \frac{0.2 * 10^{-3}}{0.5} = 4 * 10^{-4}$$

2.1 Normal Strain: Examples

a. Determine the expression for the average extensional strain in rod BC as a function of θ for $0 \leq \theta \leq \pi/2$

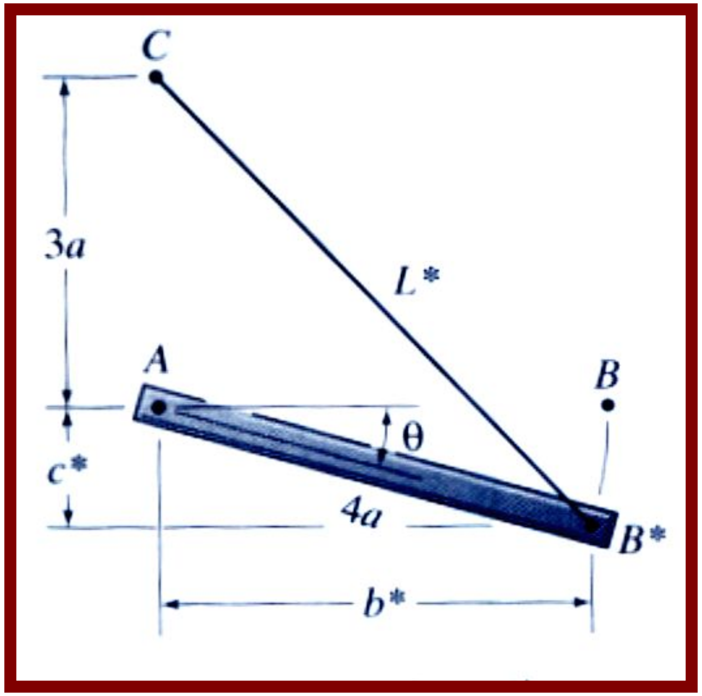
b. Determine the approximation for $\varepsilon(\theta)$ that gives acceptable accuracy for ε when $\theta << 1$ rad



When the "rigid" beam AB is horizontal, the rod BC is strain free.

2.1 Normal Strain: Examples

Deformation Diagram

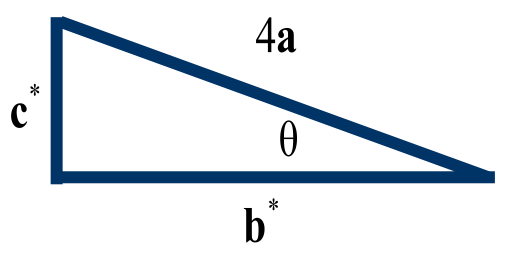


$$\varepsilon = \frac{L' - L}{L} \rightarrow \; L = \overline{BC} = 5a$$

$$L' = \sqrt{\left(3a + c^*\right)^2 + \left(b^*\right)^2} \rightarrow$$

$$b^* = 4a\cos\theta \text{ and}$$

$$c^* = 4a\sin\theta$$



2.1 Normal Strain: Examples

$$\varepsilon = \frac{L' - L}{L}$$

$$\varepsilon = \frac{\sqrt{\left(3a + 4a\sin\theta\right)^2 + \left(4a\cos\theta\right)^2} - 5a}{5a}$$

$$\varepsilon\left(\theta\right) = \frac{\sqrt{\left(3 + 4\sin\theta\right)^2 + \left(4\cos\theta\right)^2} - 5}{5}$$

$$\varepsilon\left(\theta\right) = \frac{\sqrt{9 + 24\sin\theta + 16\sin^2\theta + 16\cos^2\theta} - 5}{5}$$

$$\varepsilon\left(\theta\right) = \frac{\sqrt{25 + 24\sin\theta} - 5}{5} = \sqrt{1 + \frac{24}{25}\sin\theta} - 1$$

2.1 Normal Strain: Examples

$$\varepsilon(\theta) = \sqrt{1 + \frac{24}{25}\sin\theta} - 1$$

$$\theta << 1$$

$$\textit{Small angle approximation}: \sin\theta \approx \theta$$

$$\varepsilon(\theta) = \sqrt{1 + \frac{24}{25}\theta} - 1$$

$$\textit{By Binomial Theorem}: \sqrt{1+\beta} \approx 1 + \frac{\beta}{2}$$

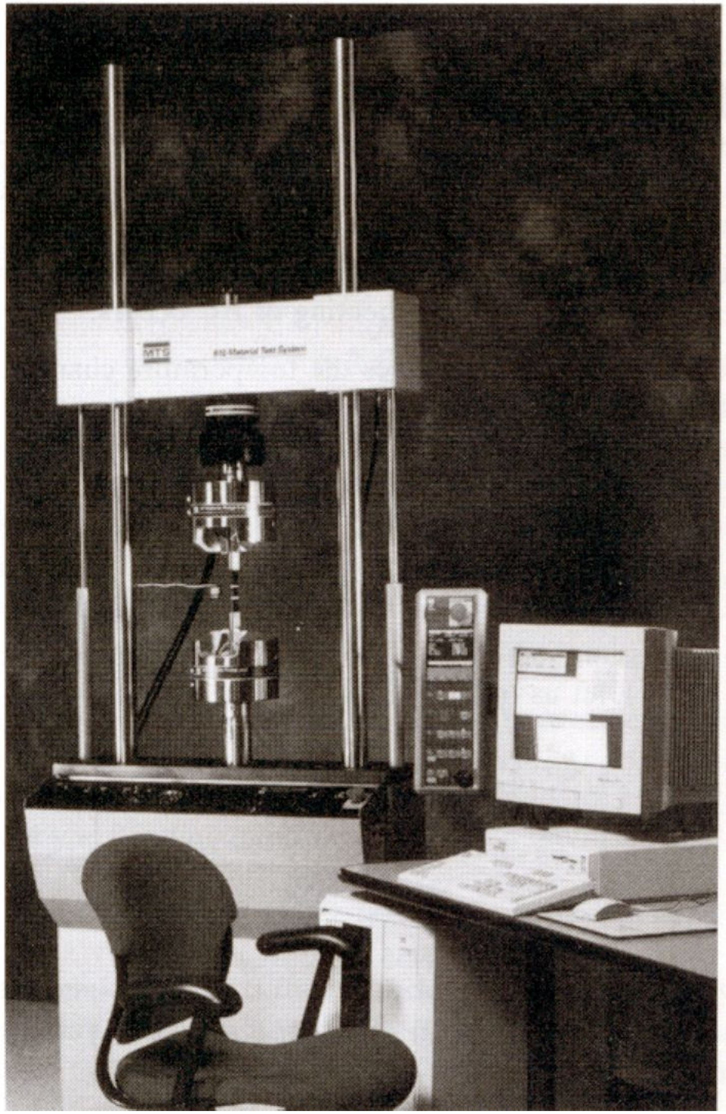
$$\therefore \varepsilon(\theta) \approx \frac{12}{25}\theta$$

The strain is dimensionless, as it should be. At $\theta = \pi/2$, $\varepsilon(\pi/2) = 2/5$. At this point $L^* = 3a + 4a = 7a$ so this value is correct.

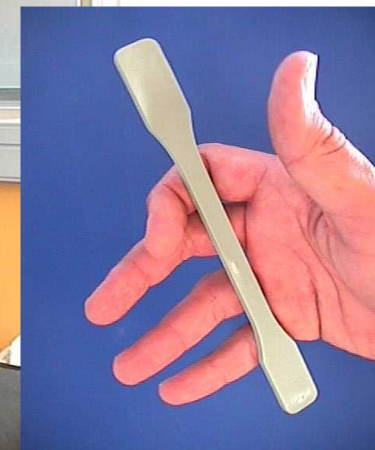
Mechanical Properties of Materials

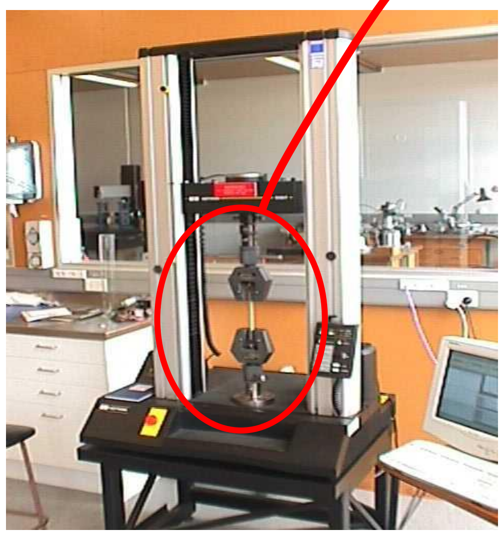
Properties are determined by mechanical tests (Tension and Compression.)

A typical test apparatus is shown on the right.



2.1 Stress Strain Diagram

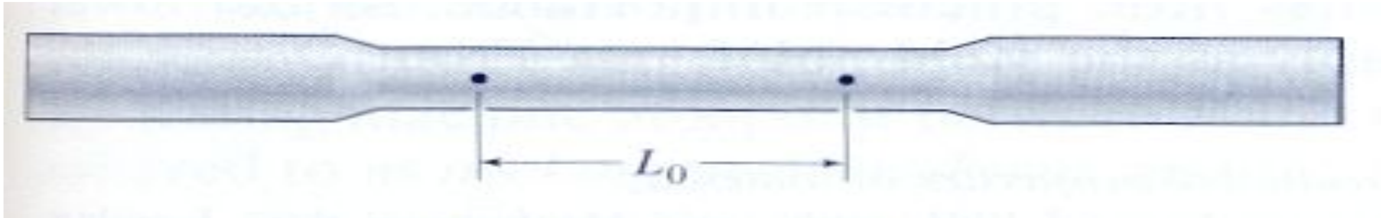




A variety of testing machine types, and sizes…

…and a variety of samples sizes.

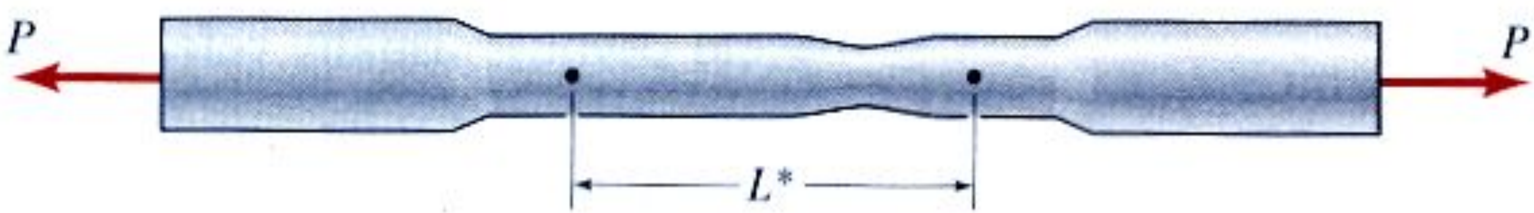
Gage Length



(a) Undeformed specimen.

Original gage length is L_0. This is not the total length of the specimen.

Deformed Specimen



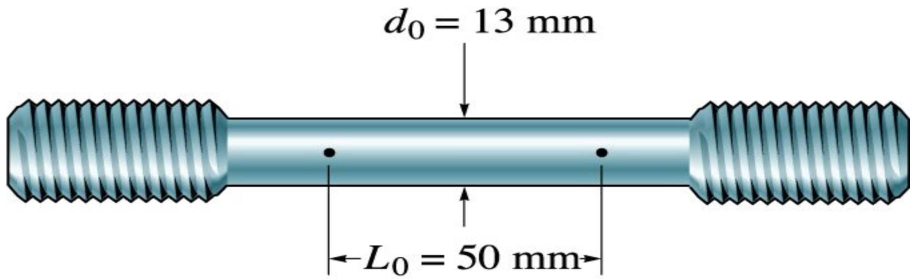
(b) Deformed specimen.

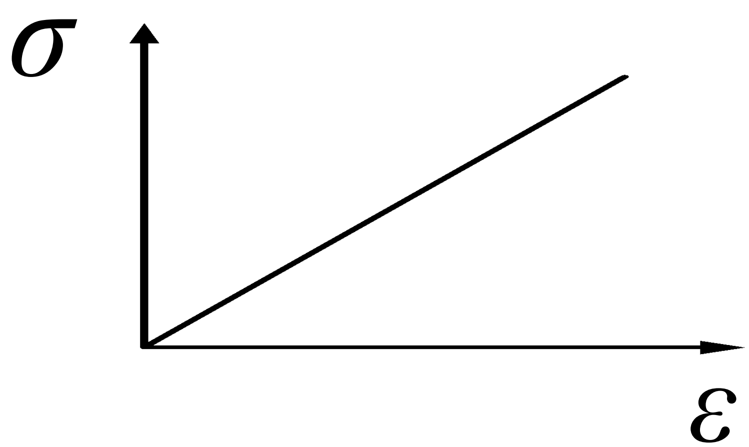
Original gage length is deformed to L^*. The load and the elongation are carefully measured. The load is slowly applied. This is a static tension test.

2.1 Stress Strain Diagram

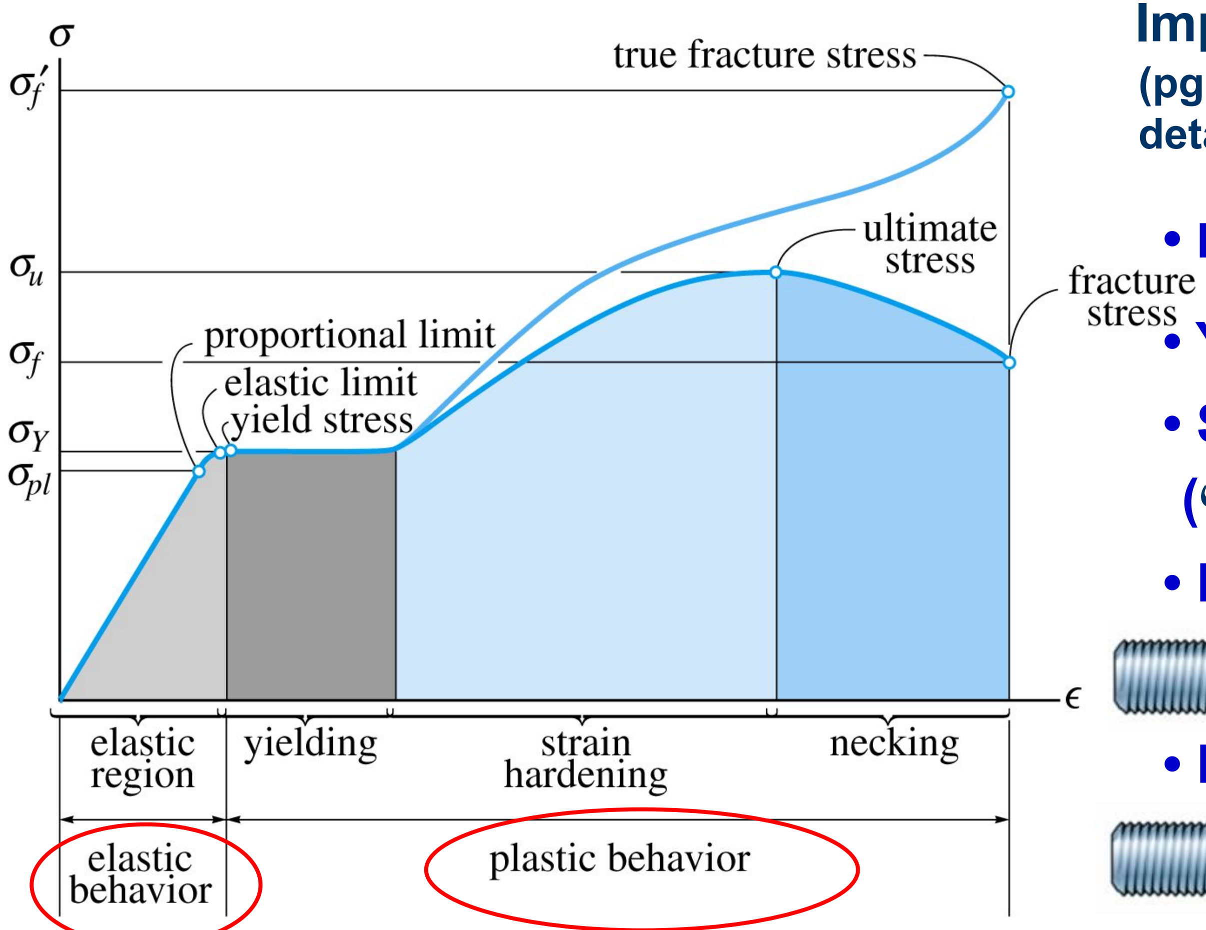
A plot of stress versus strain is called a stress strain diagram. From this diagram we can find a number of important mechanical properties.

$$\sigma = \frac{P}{A} \qquad \varepsilon = \frac{\delta_n}{L}$$





2.1 Stress Strain Diagram(Steel)



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

Recoverable(يسترد) deformation

Permanent deformation(دائم)

Important Regions:
(pg 86-88 Hibbeler gives detailed description)

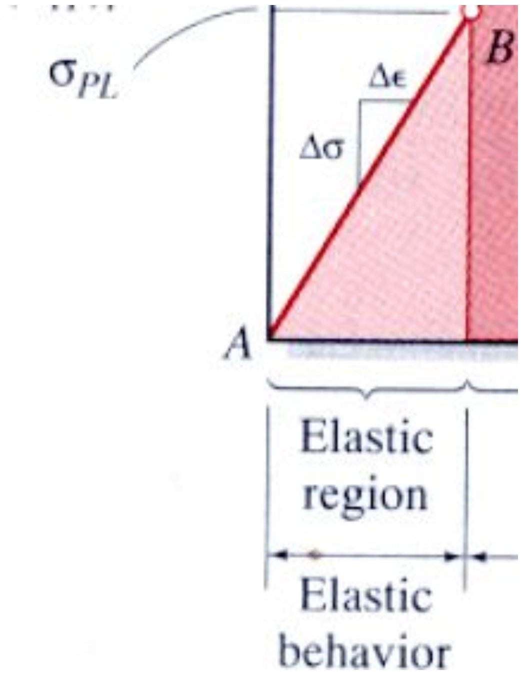
- **Elastic region(متمغّط)**
- **Yielding(مرن)**
- **Strain Hardening (تصلب)**
- **Necking(خِنَاق)**

- **Fracture(كسر)**

Note: Very little difference between **engineering** and **true** values in elastic region

2.1 Stress Strain Diagram(**Steel**)

In the figure above the region from A to B has a linear relationship between stress and strain. The stress at point B is called the proportional limit, σ_{PL}. The ratio of stress to strain in the linear region is called ***E***, the ***Young's modulus*** or the ***modulus of elasticity.***

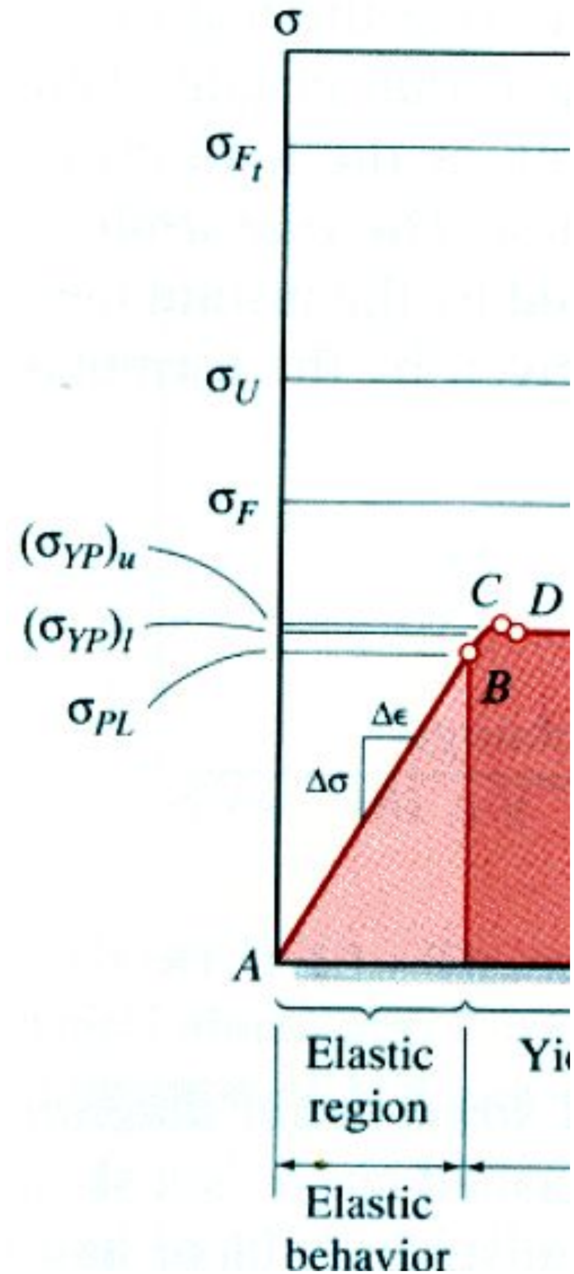


$$E = \frac{\Delta\sigma}{\Delta\varepsilon}$$

$$\sigma < \sigma_{PL}$$

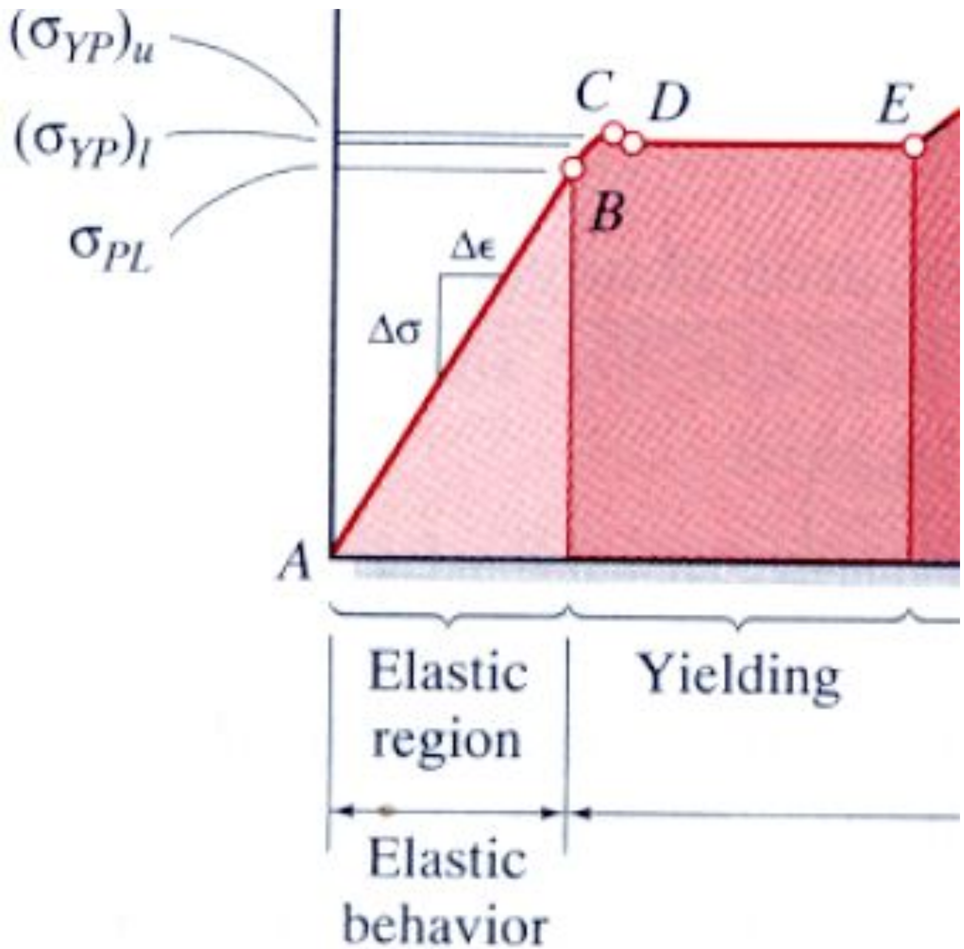
Yielding

At point B, the specimen begins yielding. Smaller load increments are required to to produce a given increment of elongation. The stress at C is called the upper yield point, $(\sigma_{YP})_u$ The stress at D is called the lower yield point, $(\sigma_{YP})_l$ The upper yield point is seldom used and the lower yield point is often referred to simply as the yield point, σ_{YP}



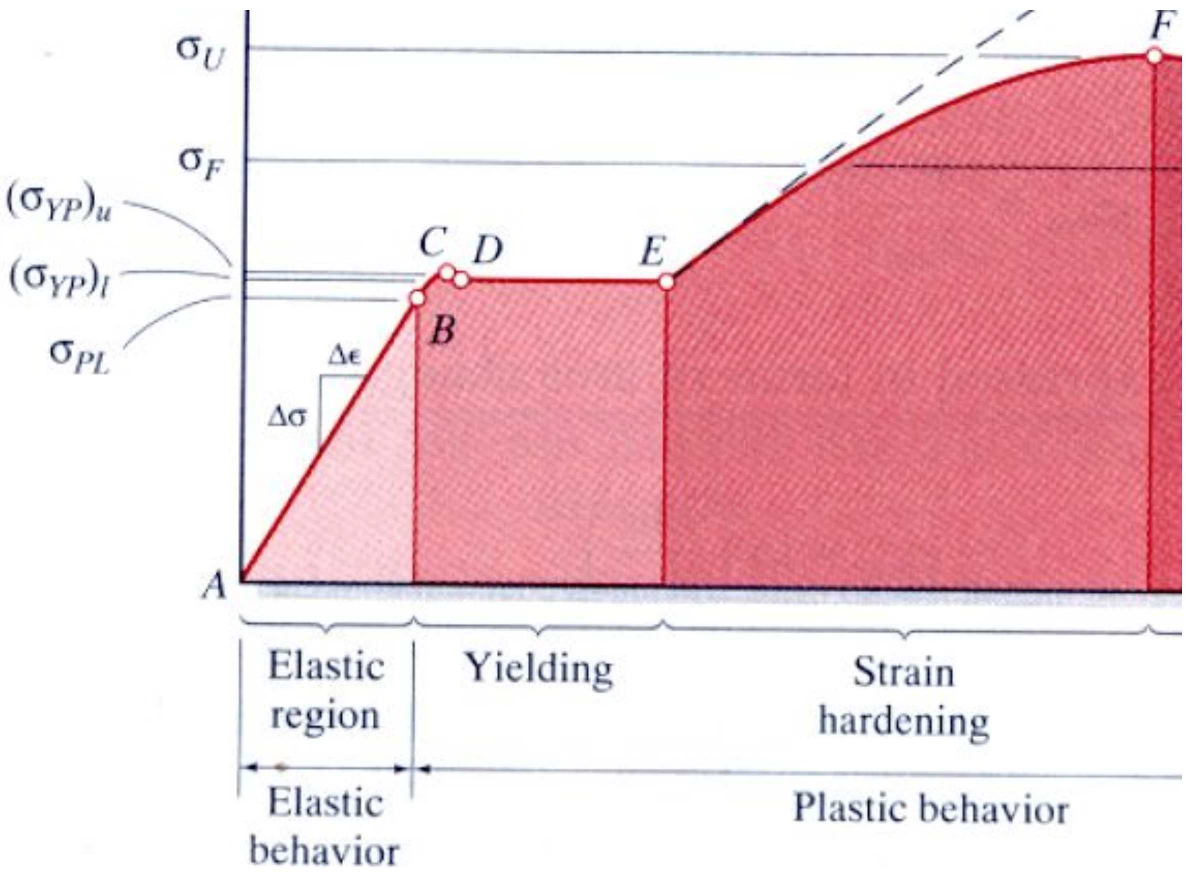
Perfectly Plastic Zone

From D to E the specimen continues to elongate without any increase in stress. The region DE is referred to as the perfectly plastic zone.



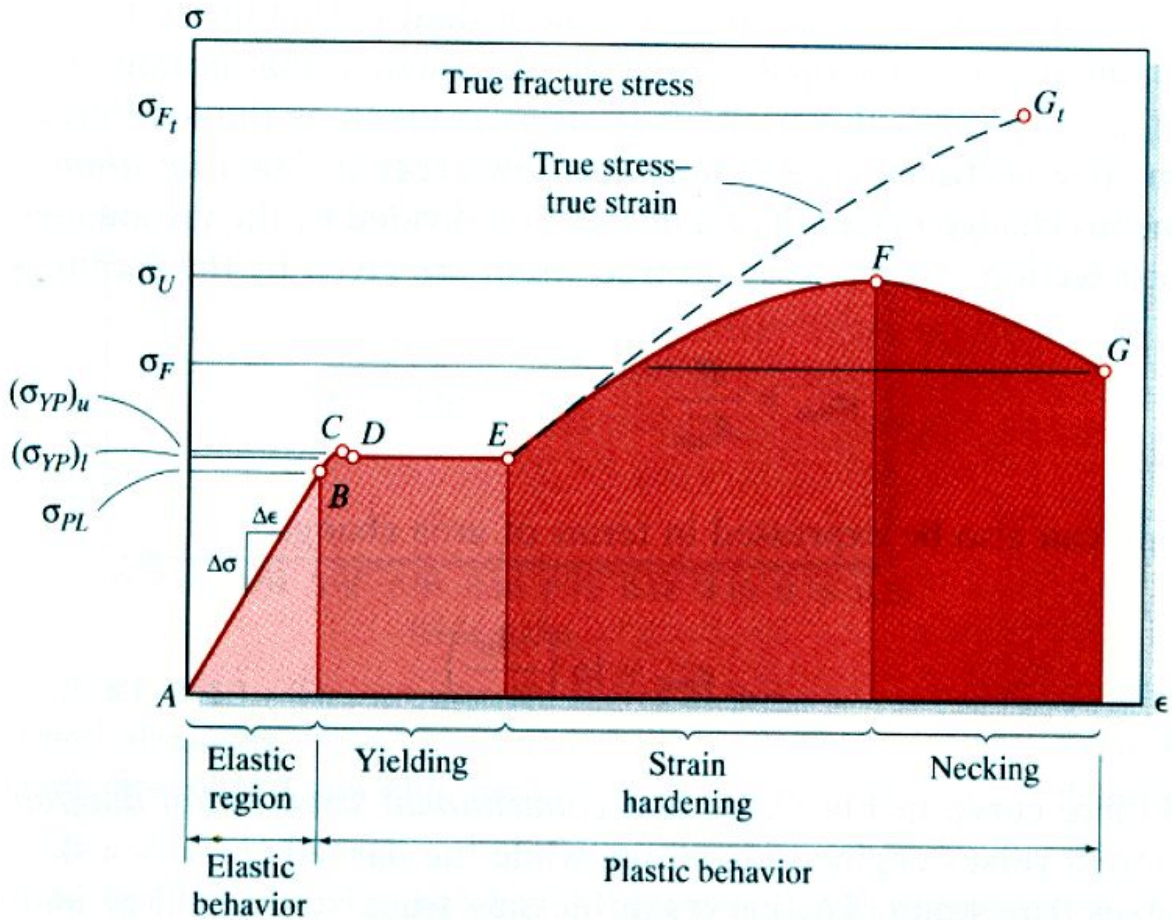
Strain Hardening

The stress begins to increase at E. The region from E to F is known as the strain hardening zone. The stress at F is the ultimate stress.

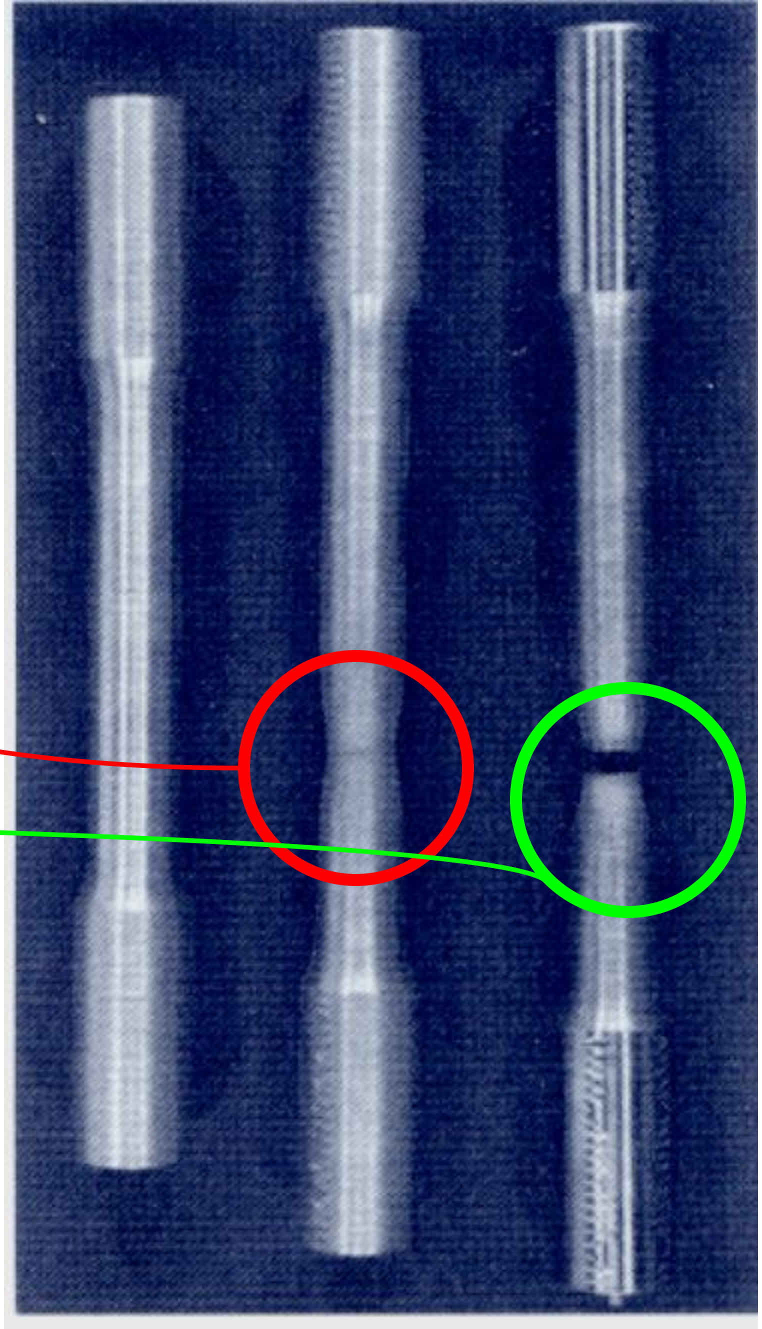


Necking

At F the stress begins to drop as the specimen begins to "neck down." This behavior continues until fracture occurs at point G, at the fracture stress, σ_F



(a)



Necking

Fracture

True Stress

Use the current minimum area rather than the original area :

$$\sigma_{true} = \frac{P}{A_{min}}$$

True Strain

Using all of the successiveمتعاقب values of L that they have recorded. Dividing the increment dL of the distance between the gage marks, by the corresponding value of L. (sum of the incremental elongations divided by the current gauge length)

$$\varepsilon_{true} = \sum \Delta\varepsilon = \sum \frac{\Delta L}{L}$$

$$\Rightarrow \varepsilon_{true} = \int_{L_0}^{L} \frac{dL}{L} = \ln(1+\varepsilon)$$

$$or$$

$$\varepsilon_{true} = \ln\left(\frac{A_0}{A_{\min}}\right)$$

Design Properties

1. ***Strength***
2. ***Stiffness***
3. ***Ductility***

Strength

Yield Strength: Highest stress that the material can withstand يقاوم without undergoing significant yielding and permanent deformation.

$$\sigma_Y = \sigma_{YP} \text{ or } \sigma_Y = \sigma_{YS}$$

Ultimate Strength: Highest value of stress (maximum value of engineering stress) that the material can withstand.

$$\sigma_{\mathbf{U}}$$

Fracture Stress: The value of stress at fracture.

$$\sigma_{\mathbf{F}}$$

Stiffness

The ratio of stress to strain (or load to displacement.) Generally of interest in the linear elastic range. The Young's modulus or modulus of elasticity, E, is used to represent a material's stiffness.

Ductilityتمدد

1. Materials that can undergo a large strain before fracture are classified as ***ductile materials***.

2. Materials that fail at small values of strain are classified as ***brittle materials***.

3. Really referring to modes of fracture.

Ductility Measures

% Elongation

The final elongation expressed as a percentage of the original gage length :

$$\text{Percent Elongation} = \left(\frac{L_F - L_0}{L_0}\right) \times 100\%$$

% Reduction in Area

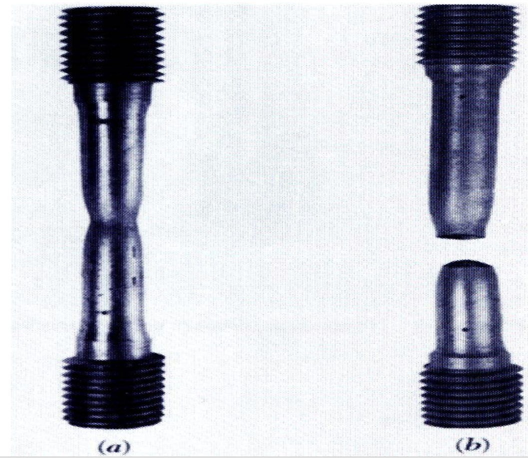
$$\text{Percent Reduction in Area} = \left(\frac{A_0 - A_F}{A_0}\right) \times 100\%$$

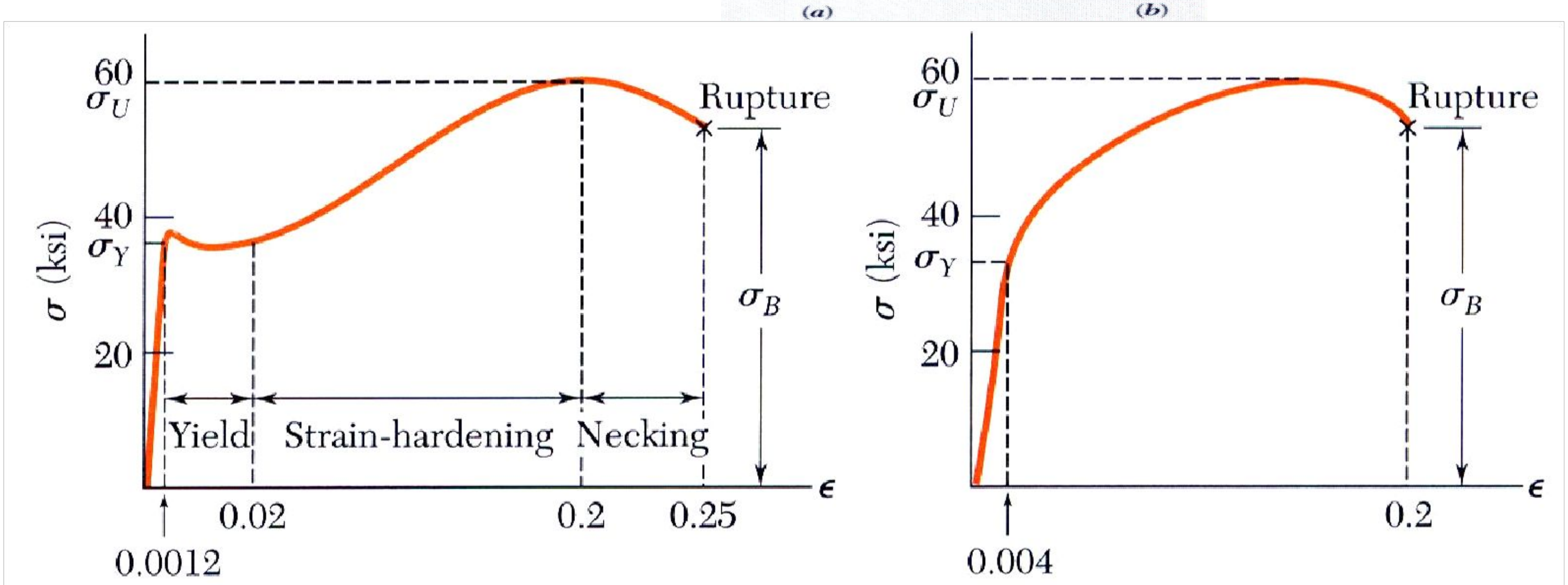
Ductile Materials

1. Steel
2. Brass
3. Aluminum
4. Copper
5. Nickel
6. Nylon

2.1 Stress Strain Diagram

Ductile Materials(لَدْن)

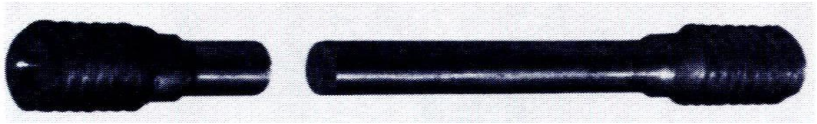




(*a*) Low-carbon steel (*b*) Aluminum alloy

2.1 Stress Strain Diagram

Brittle Materials(هش)



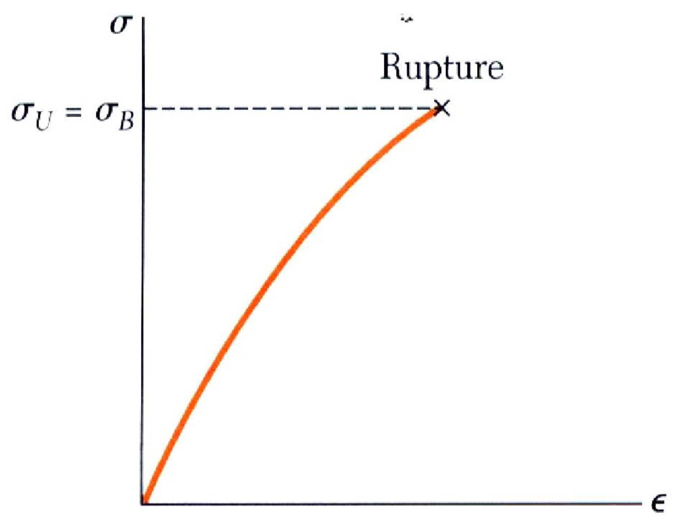
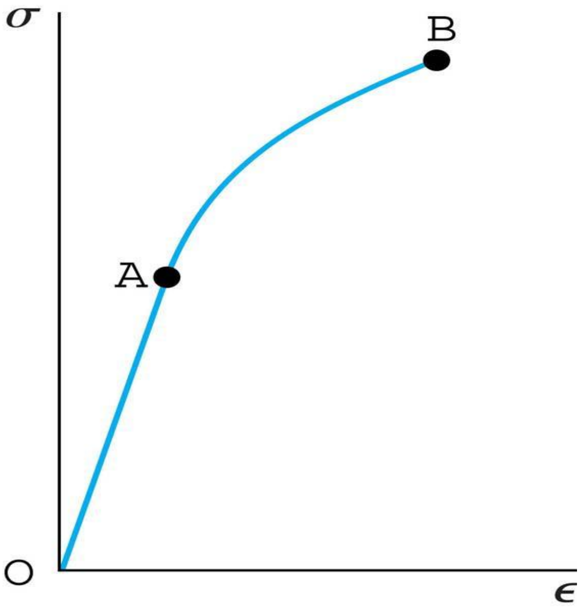


Fig. 2.11 Stress-strain diagram for a typical brittle material.



Typical stress-strain diagram for a brittle material showing the proportional limit (point *A*) and fracture stress (point *B*)
No yielding, or necking is evident. For brittle materials that fail the pieces still fit together e.g. glass or ceramics.

2.1 Stress Strain Diagram
Elastic versus Plastic Behavior

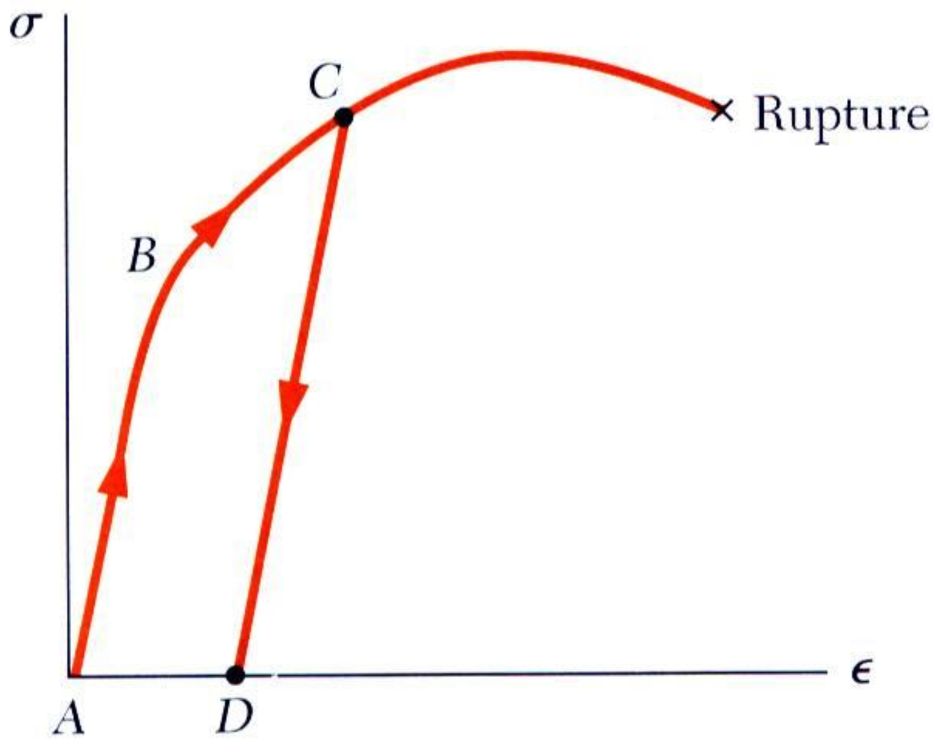
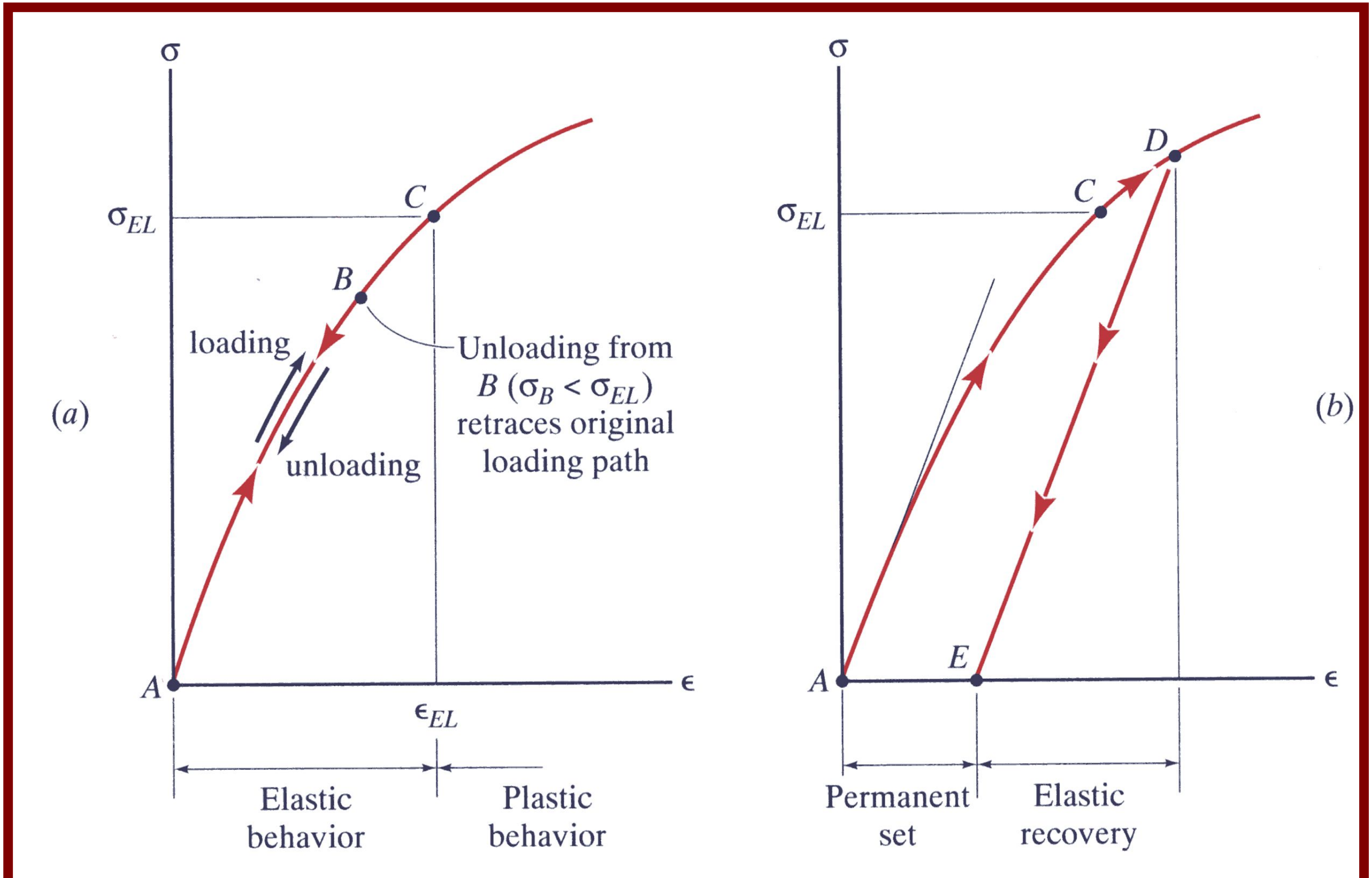
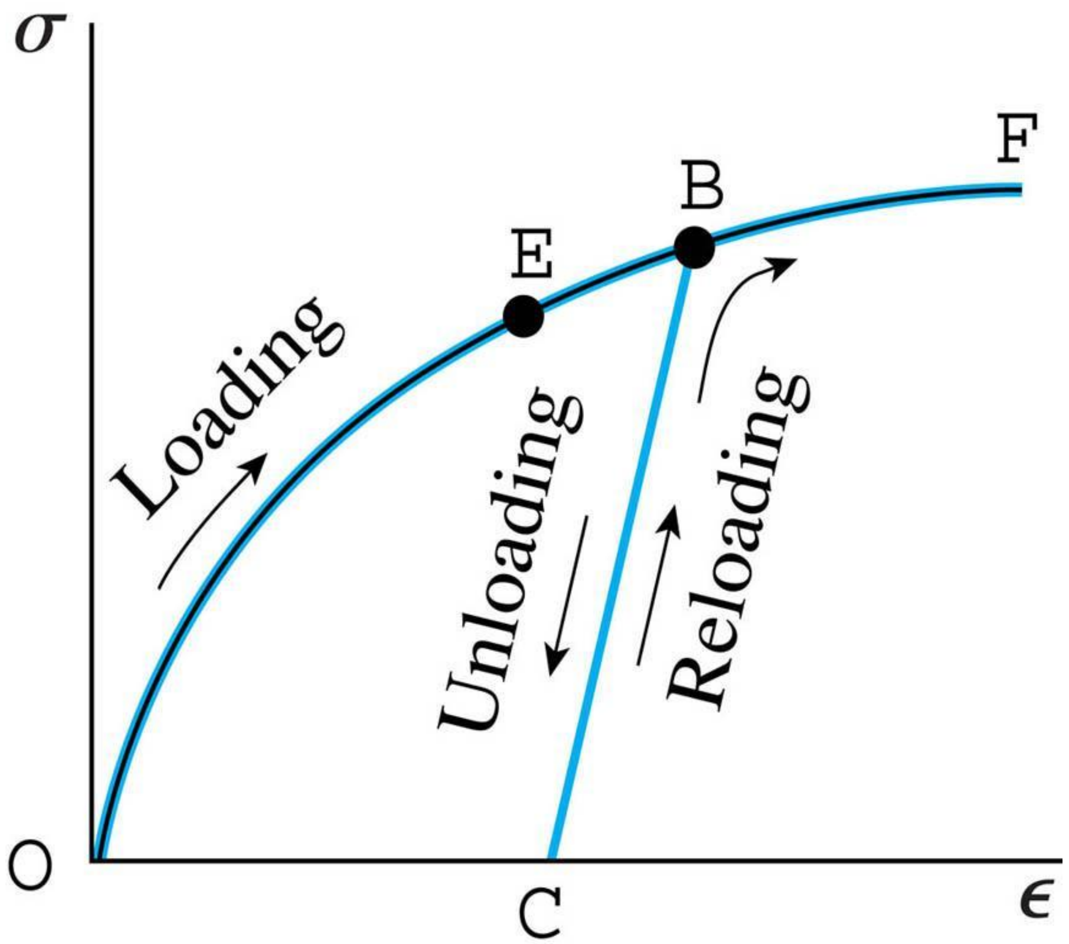


Fig. 2.18

- If the strain disappears when the stress is removed, the material is said to behave *elastically*.
- The largest stress for which this occurs is called the *elastic limit*.
- When the strain does not return to zero after the stress is removed, the material is said to behave *plastically*.

Plastic Behavior





Mechanical properties depend on the history of the piece.

After reloading of a piece the elastic and proportional limit can be increased.

2.2 Hooke's Low: Modulus of elasticity

- Below the yield stress

$$\sigma = E\,\varepsilon$$

E = Youngs Modulus or

Modulus of Elasticity

- Strength is affected by(مُتَأَثِّر) alloying(خَلِيط), heat treating, and manufacturing process but stiffness (Modulus of Elasticity) is not.

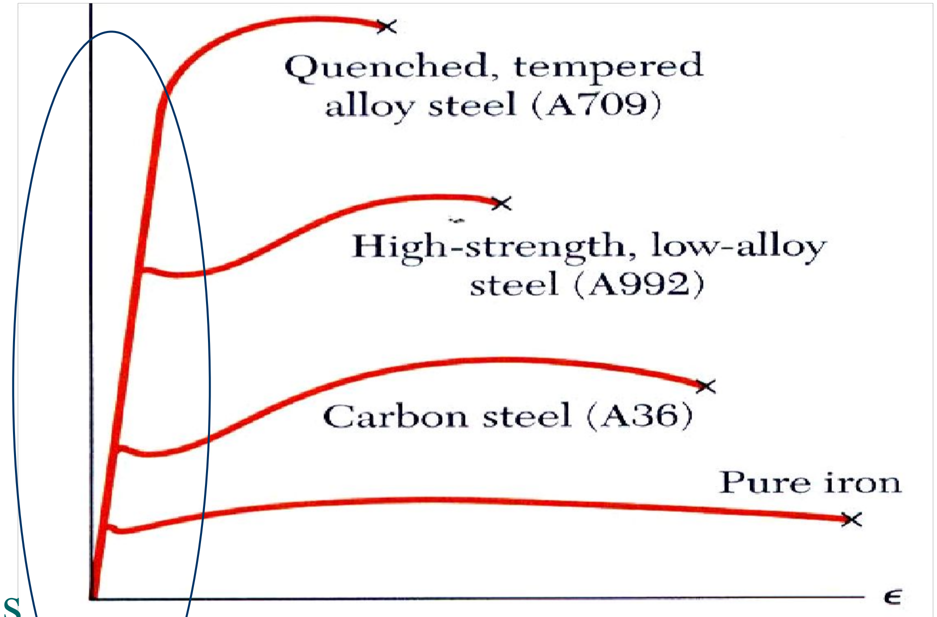


Fig. 2.16 Stress-strain diagrams for iron and different grades of steel.

$$\sigma(\varepsilon) = a * \varepsilon + b$$

$$\sigma(\varepsilon)_{\varepsilon=\varepsilon_0} = \sigma_0 \quad and \quad \sigma(\varepsilon)_{\varepsilon=\varepsilon_e} = \sigma_e \Rightarrow$$

$$\sigma_0 = a * \varepsilon_0 + b \quad and \quad \sigma_e = a * \varepsilon_e + b$$

$$a = \frac{\sigma_e - \sigma_0}{\varepsilon_e - \varepsilon_0} = \tan(\alpha) = E \quad bei \quad b = 0$$

2.8 Deformations Under Axial Loading

- From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

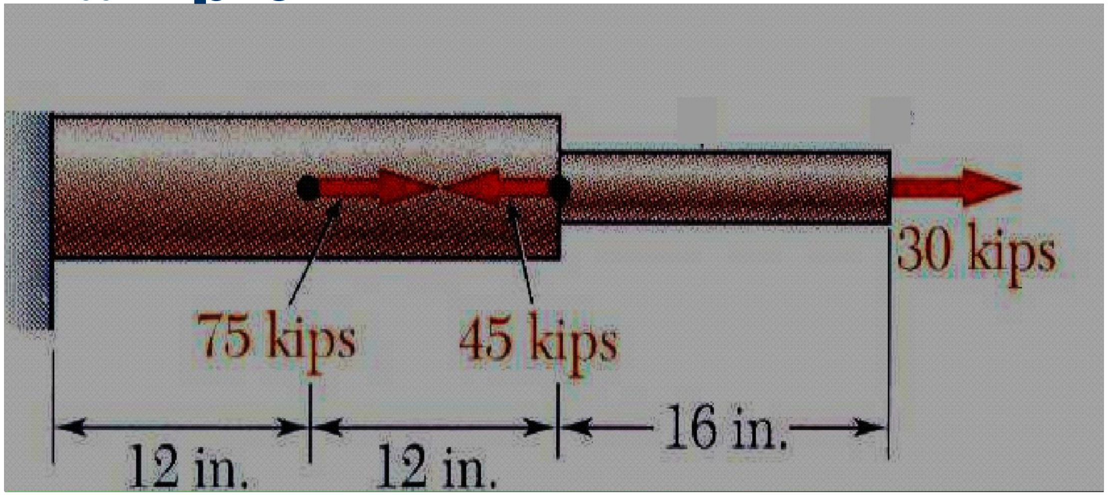
- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

2.8 **Deformation under Axial Loading Example**



$$E = 29\times10^{-6}\,\text{psi}$$

$$D = 1.07\text{ in.} \quad d = 0.618\text{ in.}$$

Determine the deformation of the steel rod shown under the given loads.

$$\delta = 75.9\times10^{-3}\text{ in.}$$

SOLUTION:

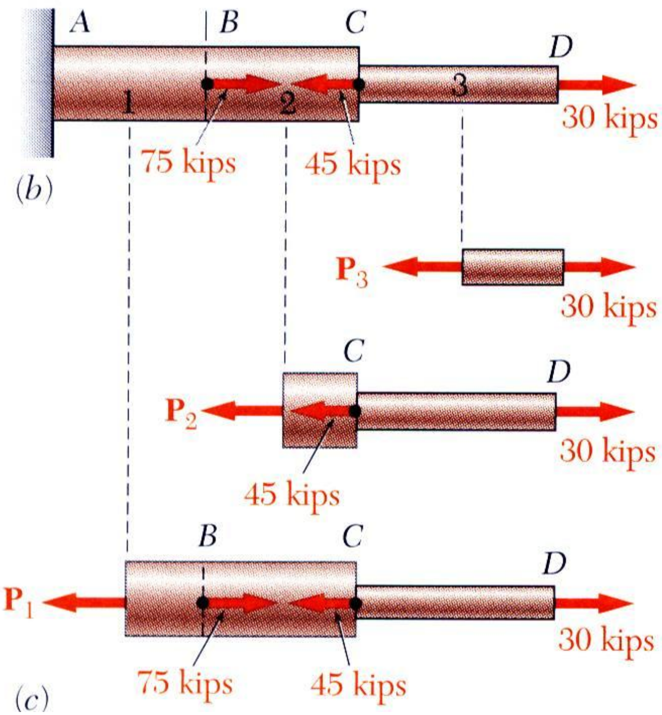
- Divide the rod into components at the load application points.
- Apply a free-body analysis on each component to determine the internal force
- Evaluate the total of the component deflections.

2.8 **Deformation under Axial Loading**

Example

SOLUTION:

- Divide the rod into three components:



$L_1 = L_2 = 12\text{ in.}$ $\quad L_3 = 16\text{ in.}$

$A_1 = A_2 = 0.9\text{ in}^2$ $\quad A_3 = 0.3\text{ in}^2$

- Apply free-body analysis to each component to determine internal forces,

$$P_1 = 60 \times 10^3\text{ lb}$$

$$P_2 = -15 \times 10^3\text{ lb}$$

$$P_3 = 30 \times 10^3\text{ lb}$$

- Evaluate total deflection,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$

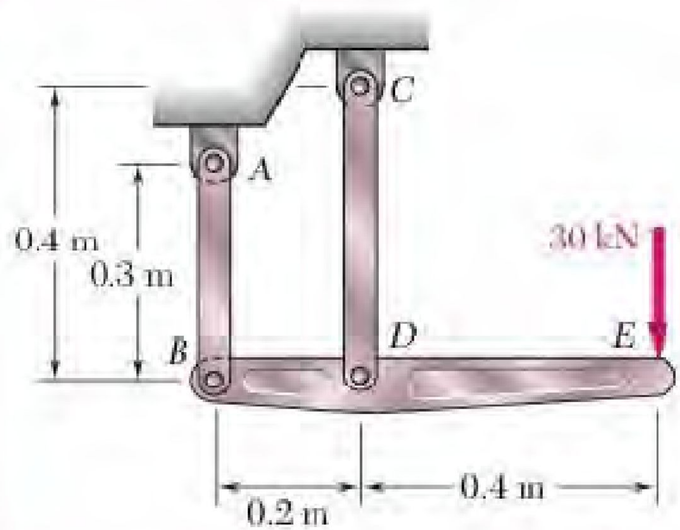
$$= \frac{1}{29 \times 10^6}\left[\frac{\left(60 \times 10^3\right)12}{0.9} + \frac{\left(-15 \times 10^3\right)12}{0.9} + \frac{\left(30 \times 10^3\right)16}{0.3}\right]$$

$$= 75.9 \times 10^{-3}\text{in.}$$

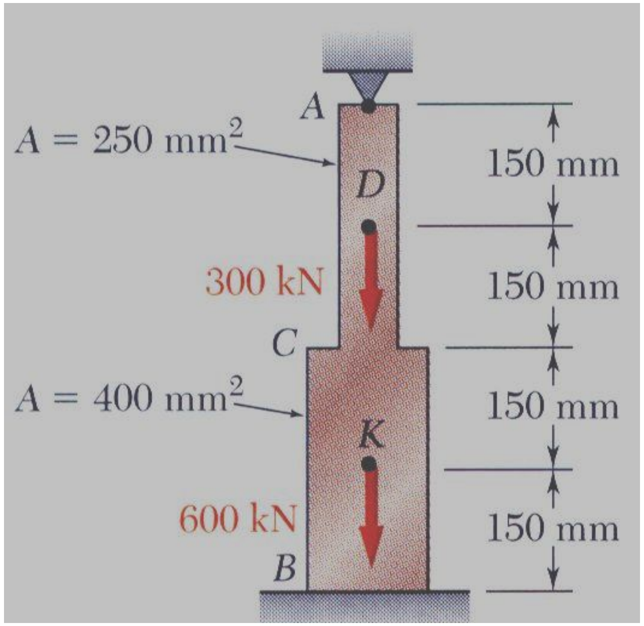
$$\boxed{\delta = 75.9 \times 10^{-3}\text{ in.}}$$

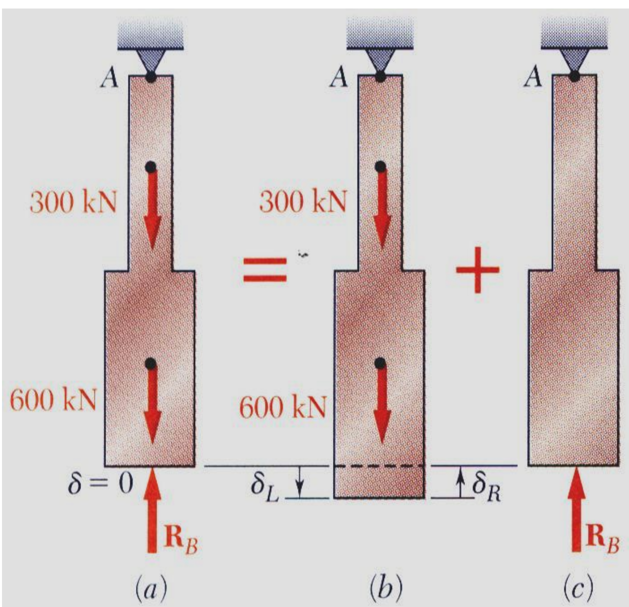
SAMPLE PROBLEM 2.1

The rigid bar *BDE* is supported by two links *AB* and *CD*. Link *AB* is made of aluminum ($E = 70$ GPa) and has a cross-sectional area of 500 mm^2; link *CD* is made of steel ($E = 200$ GPa) and has a cross-sectional area of 600 mm^2. For the 30-kN force shown, determine the deflection (*a*) of *B*, (*b*) of *D*, (*c*) of *E*.



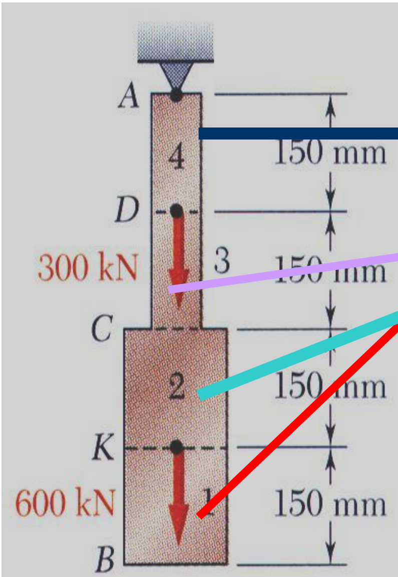
2.9 Static Indeterminacy





- Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.
- A structure will be statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.
- Redundant reactions are replaced with unknown loads which along with the other loads must produce compatible deformations.
- Deformations due to actual loads and redundant reactions are determined separately and then added or *superposed*.

$$\delta = \delta_L + \delta_R = 0$$



SOLUTION:

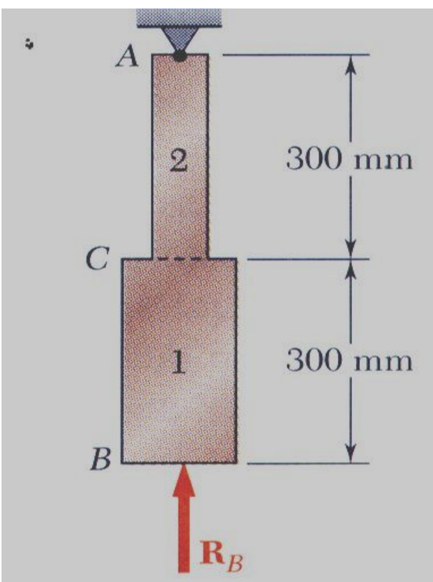
- Solve for the displacement at *B* due to the applied loads with the redundant constraint released,

$$P_1 = 0 \quad P_2 = P_3 = 600\times10^3\,\mathrm{N} \quad P_4 = 900\times10^3\,\mathrm{N}$$

$$A_1 = A_2 = 400\times10^{-6}\,\mathrm{m}^2 \quad A_3 = A_4 = 250\times10^{-6}\,\mathrm{m}^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150\,\mathrm{m}$$

$$\delta_L = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1.125\times10^9}{E}$$



- Solve for the displacement at *B* due to the redundant constraint,

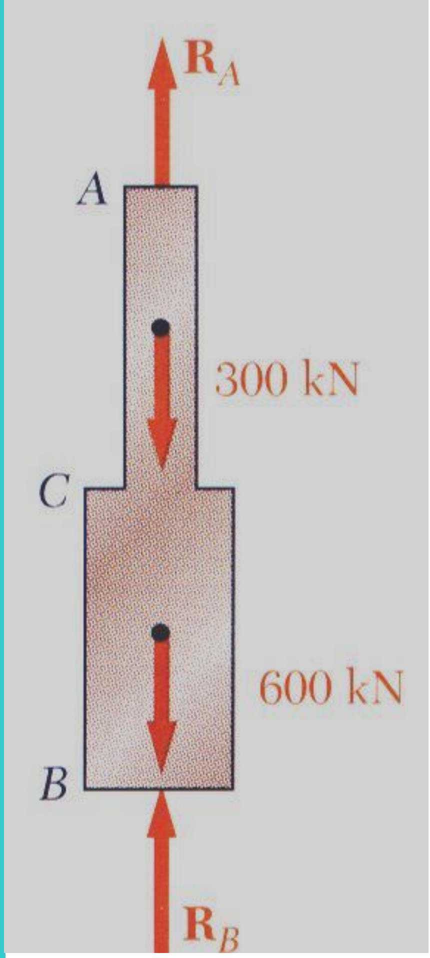
$$P_1 = P_2 = -R_B$$

$$A_1 = 400\times10^{-6}\,\mathrm{m}^2 \quad A_2 = 250\times10^{-6}\,\mathrm{m}^2$$

$$L_1 = L_2 = 0.300\,\mathrm{m}$$

$$\delta_R = \sum_i \frac{P_i L_i}{A_i E_i} = -\frac{\left(1.95\times10^3\right)R_B}{E}$$

2.9 **Static Indeterminacy**



- Require that the displacements due to the loads and due to the redundant reaction be compatible,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta = \frac{1.125\times 10^9}{E} - \frac{\left(1.95\times 10^3\right)R_B}{E} = 0$$

$$R_B = 577\times 10^3\,\mathrm{N} = 577\,\mathrm{kN}$$

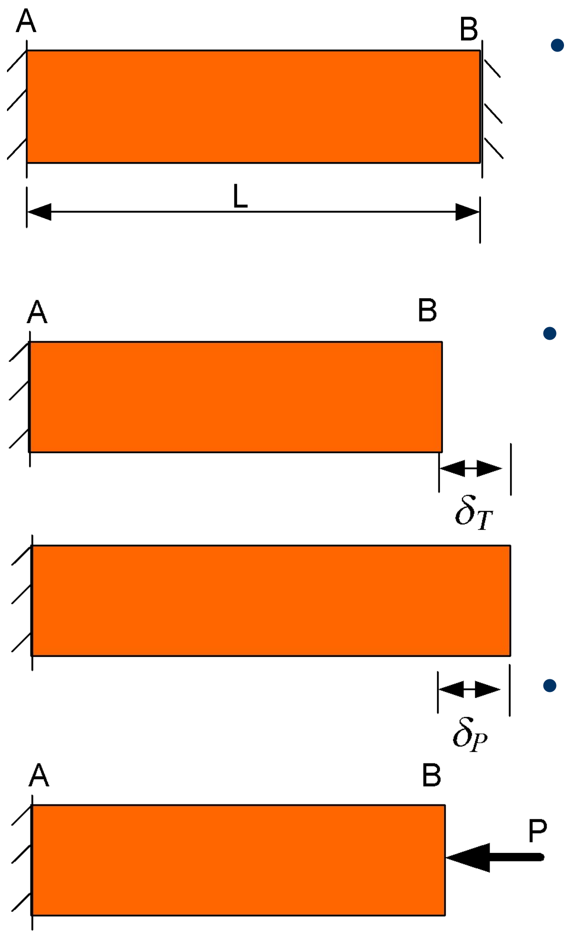
- Find the reaction at *A* due to the loads and the reaction at *B*

$$\sum F_y = 0 = R_A - 300\,\mathrm{kN} - 600\,\mathrm{kN} + 577\,\mathrm{kN}$$

$$R_A = 323\,\mathrm{kN}$$

$$R_A = 323\,\mathrm{kN}$$
$$R_B = 577\,\mathrm{kN}$$

2.10 Thermal Stresses



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.

- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$$\alpha = \text{thermal expansion coef.}$$

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

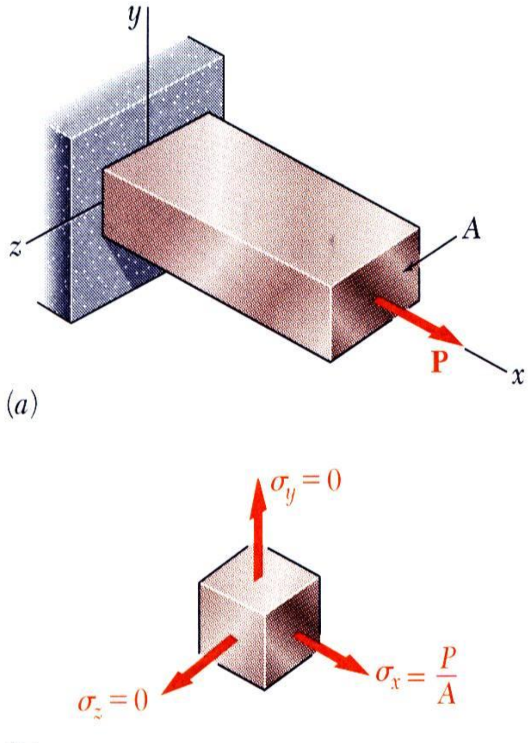
$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

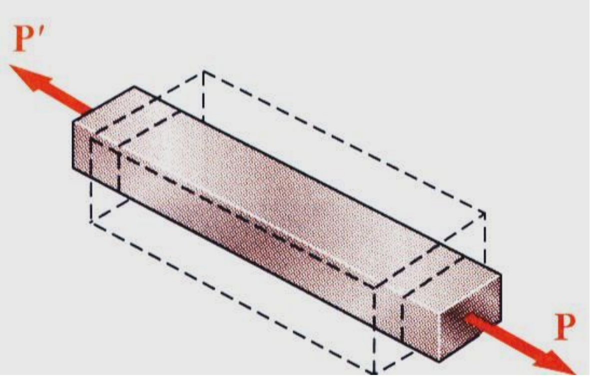
$$\delta = \delta_T + \delta_P = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

2.10 **Poisson's Ratio**





- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$

$$with \qquad \varepsilon_i = \frac{L_i - l_{i0}}{l_{i0}}$$

- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic and homogeneous (no direction and position independence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

- Poisson's ratio is defined as

$$\nu = \left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

2.10 Poisson's Ratio

Siméon Poisson

"Life is good for only two things, discovering mathematics and teaching mathematics."

$$\varepsilon_{transverse} = -\nu\, \varepsilon_{longitudinal}$$

ν (Greek letter nu) is called the Poisson's ratio. Typical values are in the 0.2 – 0.35 range.

A 500-mm-long, 16-mm-diameter rod made of a homogenous, isotropic material is observed to increase in length by 300 μm, and to decrease in diameter by 2.4 μm when subjected to an axial 12-kN load. Determine the modulus of elasticity and Poisson's ratio of the material.

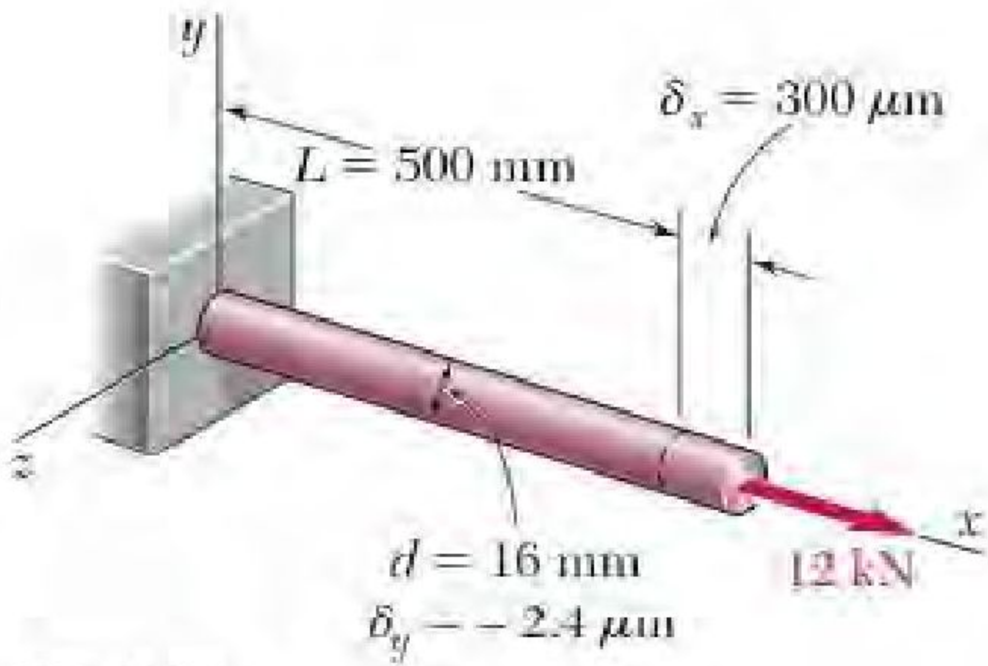
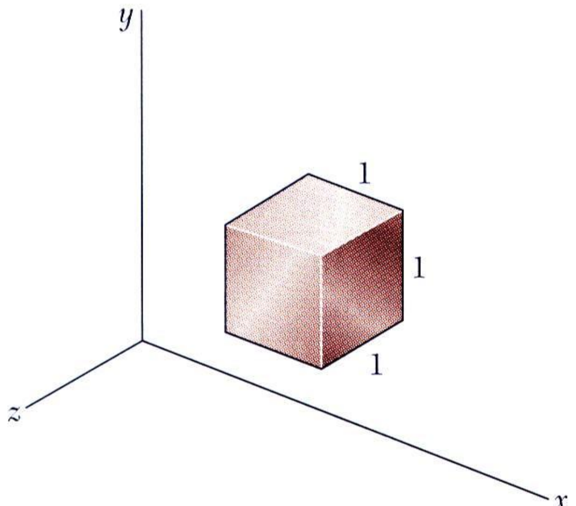
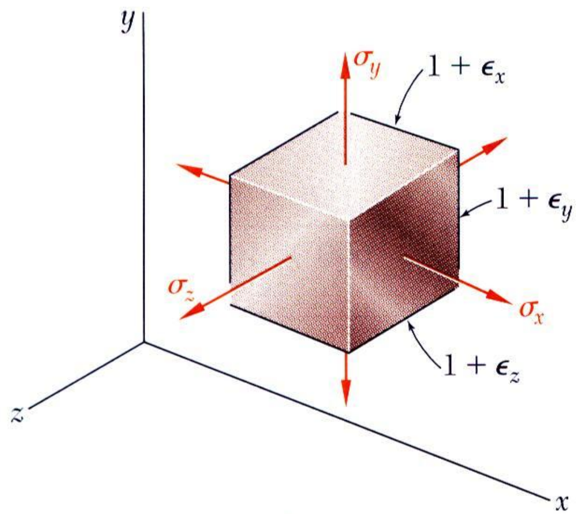


Fig. 2.41

2.11 **Generalized Hooke's Law**



(a)



(b)

- For an element subjected to multi-axial loading, the normal strain components resulting from the stress components may be determined from the *principle of superposition*. This requires:

  1)strain is linearly related to stress
  2)deformations are small

- With these restrictions:

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

The steel block shown (Fig. 2.44) is subjected to a uniform pressure on all its faces. Knowing that the change in length of edge AB is -1.2×10^{-3} in., determine (*a*) the change in length of the other two edges, (*b*) the pressure p applied to the faces of the block. Assume $E = 29 \times 10^{6}$ psi and $\nu = 0.29$.

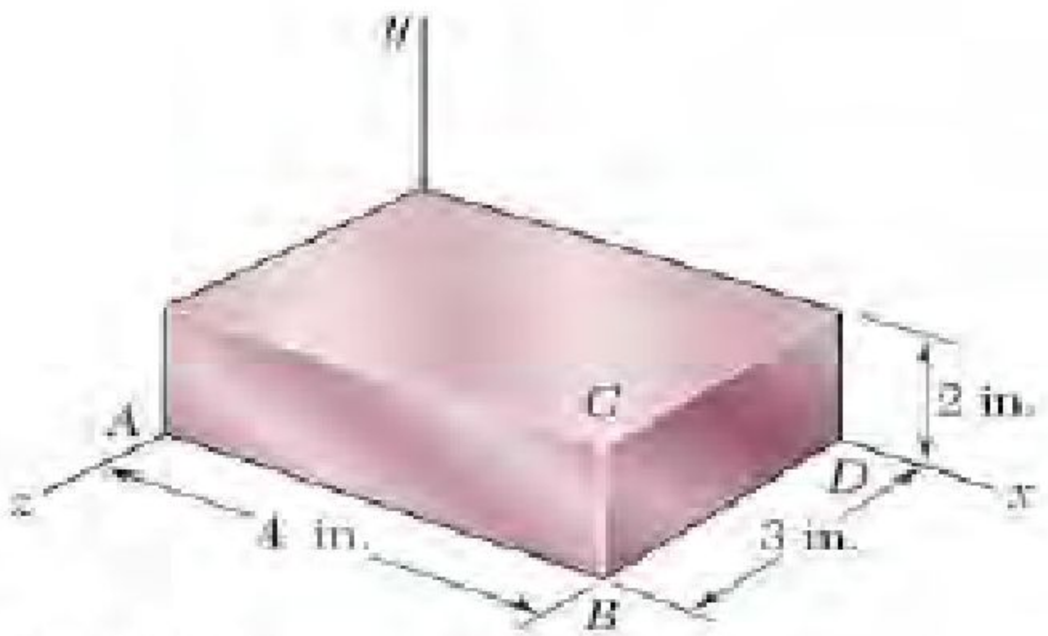
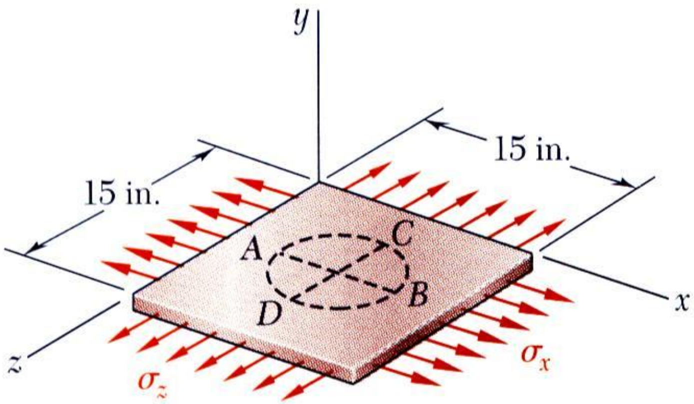


Fig. 2.44

2.11 Generalized Hooke's Law



A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = 3/4$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi.

For $E = 10\text{x}10^6$ psi and $\nu = 1/3$, determine the change in:

a) the length of diameter *AB*,

b) the length of diameter *CD*,

c) the thickness of the plate, and

d) the volume of the plate.

2.11 Relation Among *E*, *v*, and *G*

SOLUTION:

- Apply the generalized Hooke's Law to find the three components of normal strain.

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10\times 10^6\,\text{psi}}\left[(12\,\text{ksi}) - 0 - \frac{1}{3}(20\,\text{ksi})\right]$$

$$= +0.533\times 10^{-3}\,\text{in./in.}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= -1.067\times 10^{-3}\,\text{in./in.}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= +1.600\times 10^{-3}\,\text{in./in.}$$

- Evaluate the deformation components.

$$\delta_{B/A} = \varepsilon_x d = \left(+0.533\times 10^{-3}\,\text{in./in.}\right)(9\,\text{in.})$$

$$\boxed{\delta_{B/A} = +4.8\times 10^{-3}\,\text{in.}}$$

$$\delta_{C/D} = \varepsilon_z d = \left(+1.600\times 10^{-3}\,\text{in./in.}\right)(9\,\text{in.})$$

$$\boxed{\delta_{C/D} = +14.4\times 10^{-3}\,\text{in.}}$$

$$\delta_t = \varepsilon_y t = \left(-1.067\times 10^{-3}\,\text{in./in.}\right)(0.75\,\text{in.})$$

$$\boxed{\delta_t = -0.800\times 10^{-3}\,\text{in.}}$$

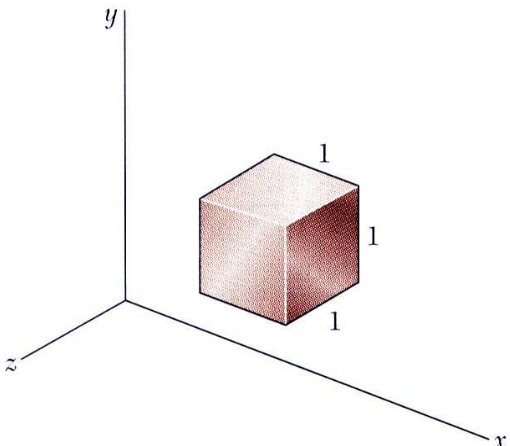
- Find the change in volume

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1.067\times 10^{-3}\,\text{in}^3/\text{in}^3$$

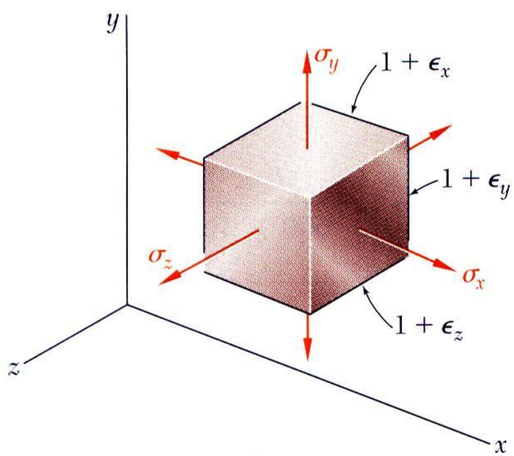
$$\Delta V = eV = 1.067\times 10^{-3}(15\times 15\times 0.75)\text{in}^3$$

$$\boxed{\Delta V = +0.187\,\text{in}^3}$$

2.11 Dilatation(اِسْتِطالة): Bulk(حجم) Modulus



(a)



(b)

- Relative to the unstressed state, the change in volume is

$$e=\left[(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)\right]-1=\left[1+\varepsilon_x+\varepsilon_y+\varepsilon_z\right]-1$$

$$=\varepsilon_x+\varepsilon_y+\varepsilon_z$$

$$=\frac{1-2\nu}{E}\left(\sigma_x+\sigma_y+\sigma_z\right)$$

= dilatation (change in volume per unit volume)

- For element subjected to uniform hydrostatic pressure,

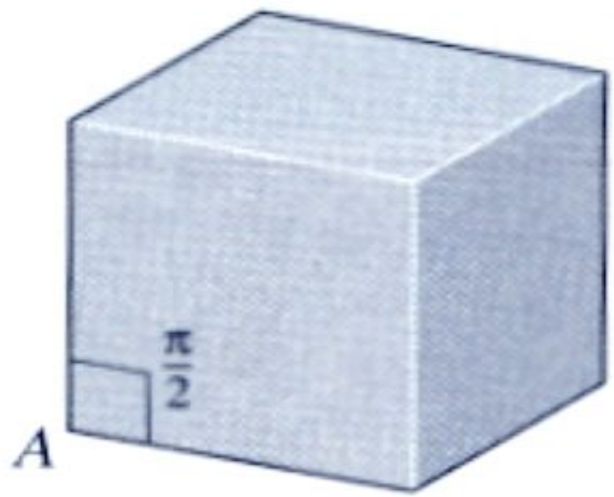
$$e=-p\frac{3(1-2\nu)}{E}=-\frac{p}{k}$$

$$k=\frac{E}{3(1-2\nu)}=\text{bulk modulus}$$

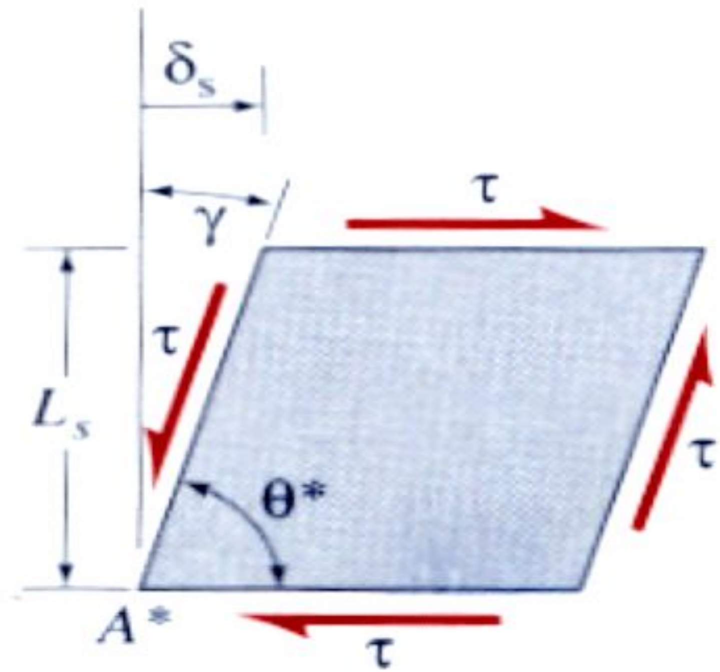
- Subjected to uniform pressure, dilatation must be negative, therefore

$$0<\nu<\frac{1}{2}$$

Shear Strain



(*a*) Original (undeformed) element.



(*b*) Pure shear deformation.

- A cubic element subjected to a shear stress will deform into a rhomboid(شبيه المعين). The corresponding *shear* strain is quantified in terms of the change in angle between the sides,

$$\tau_{xy} = f\left(\gamma_{xy}\right)$$

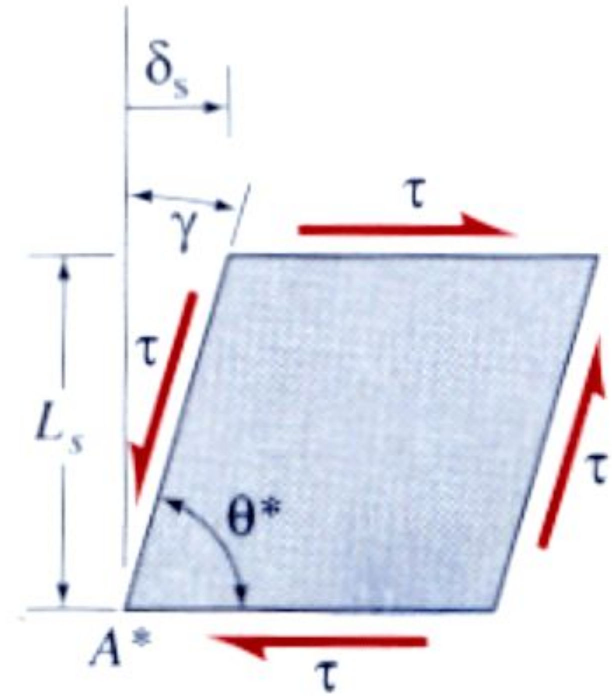
Shear Strain

$$\gamma = \frac{\pi}{2} - \theta^*$$

Small angle approximation

$$\gamma \approx \sin(\gamma) \approx \tan(\gamma)$$

$$\gamma = \frac{\pi}{2} - \theta^* \approx \tan\left(\frac{\pi}{2} - \theta^*\right) = \frac{\delta_s}{L_s}$$



(*b*) Pure shear deformation.

Hooke's Law for Shear

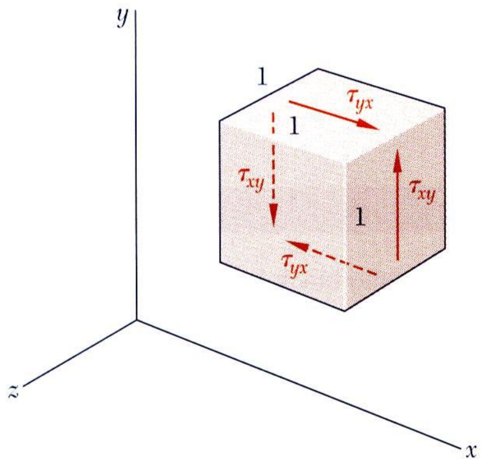


Fig. 2.46

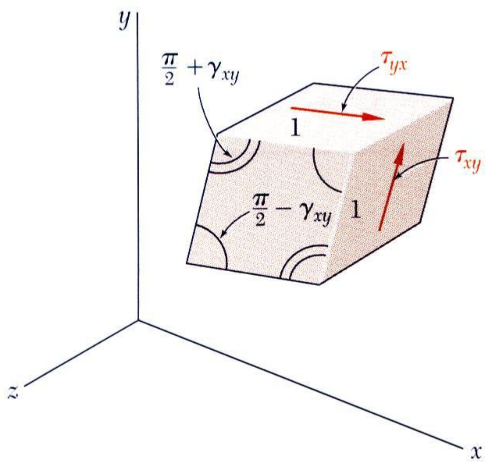


Fig. 2.47

$$\tau = G\gamma$$

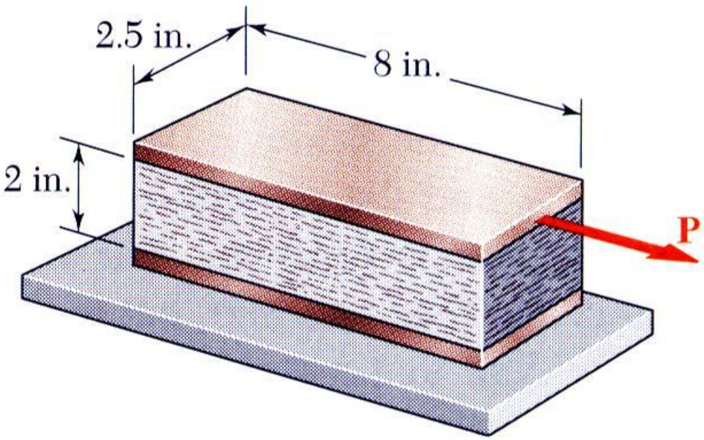
G is the shear modulus or the shear modulus of elasticity or the modulus of rigidity.

$$\tau_{xy} = G\,\gamma_{xy}$$

$$\tau_{yz} = G\,\gamma_{yz}$$

$$\tau_{zx} = G\,\gamma_{zx}$$

2.11 **Shearing Strain**

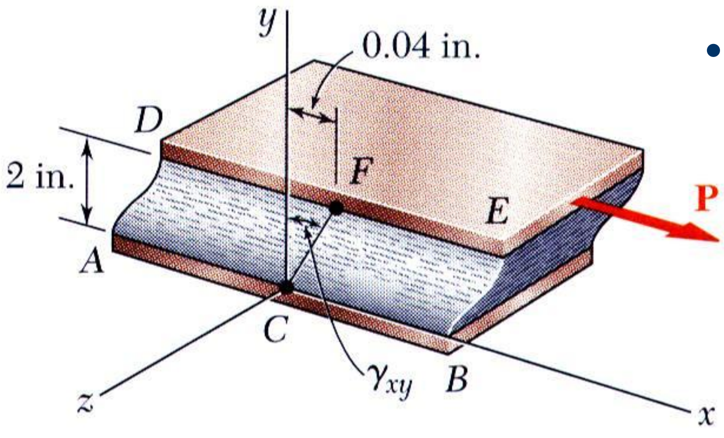


A rectangular block of material with modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force P. Knowing that the upper plate moves through 0.04 in. under the action of the force, determine a) the average shearing strain in the material, and b) the force P exerted on the plate.

SOLUTION:

- Determine the average angular deformation or shearing strain of the block.
- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.
- Use the definition of shearing stress to find the force P.

2.11 **Shearing Strain**



- Determine the average angular deformation or shearing strain of the block.

$$\gamma_{xy} \approx \tan\gamma_{xy} = \frac{0.04\,\text{in.}}{2\,\text{in.}} \qquad \gamma_{xy} = 0.020\,\text{rad}$$

- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

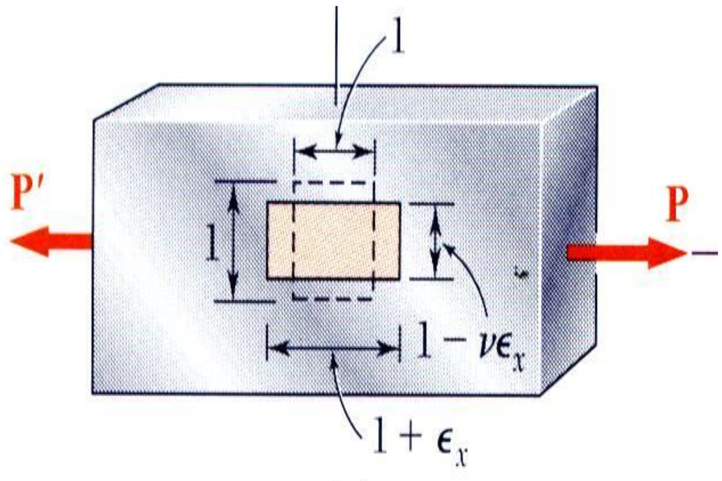
$$\tau_{xy} = G\gamma_{xy} = \left(90\times10^3\,\text{psi}\right)\left(0.020\,\text{rad}\right) = 1800\,\text{psi}$$

- Use the definition of shearing stress to find the force *P*.

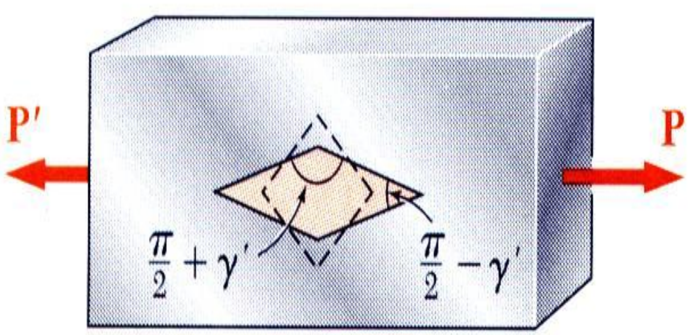
$$P = \tau_{xy}A = \left(1800\,\text{psi}\right)\left(8\,\text{in.}\right)\left(2.5\,\text{in.}\right) = 36\times10^3\,\text{lb}$$

$$\boxed{P = 36.0\,\text{kips}}$$

2.11 Relation Among *E*, *ν*, and *G*



(a)



- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1 + \nu)$$

Generalized Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

Generalized Hooke's Law

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right]$$

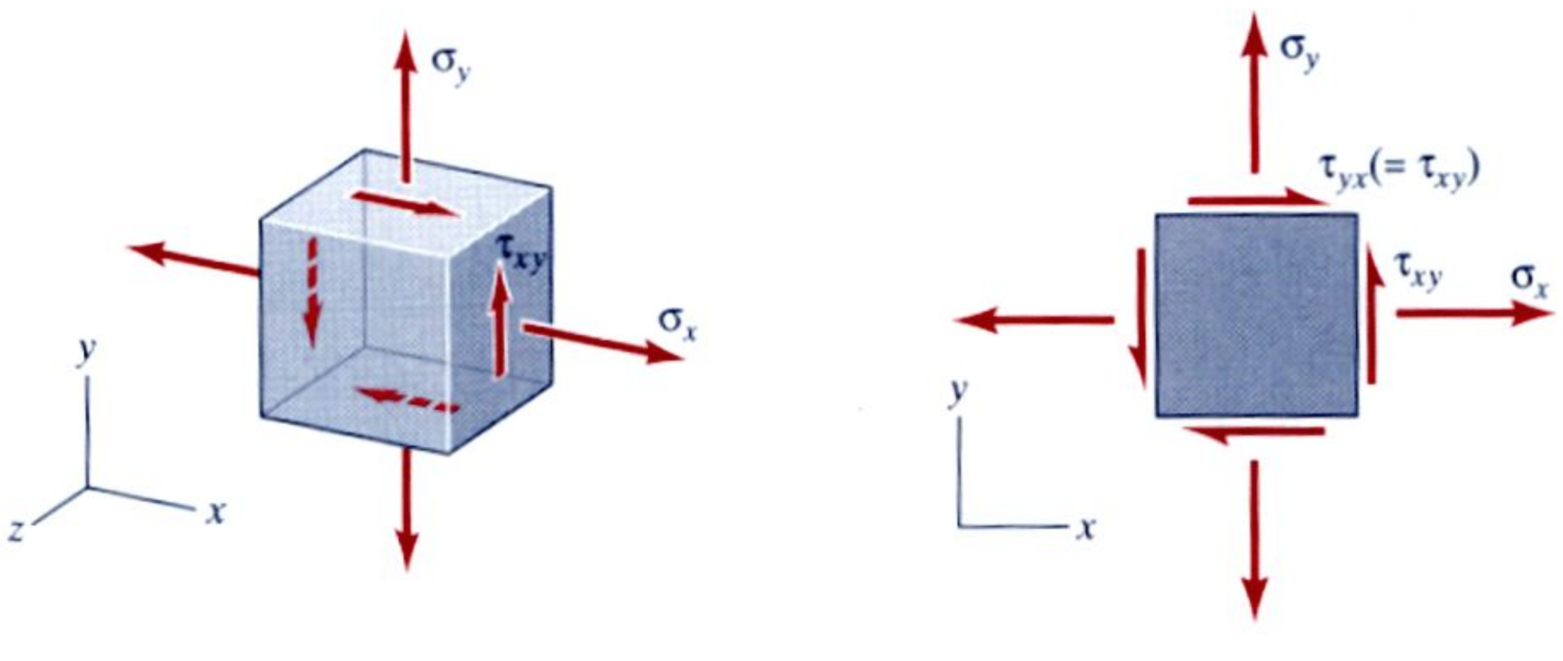
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right]$$

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{xz} = G\gamma_{xz}$$

Plane Stress

A body that is in a two-dimensional state of stress with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ is said to be in a state of plane stress.



(*a*) Three-dimensional view. (*b*) Two-dimensional view.

Generalized Hooke's Law

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \quad \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$

$$\sigma_z = \tau_{yz} = \tau_{xz} = 0 \begin{cases} \varepsilon_x = \dfrac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \\ \varepsilon_y = \dfrac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \Rightarrow \varepsilon_z = \dfrac{-\nu}{E}\left(\sigma_x + \sigma_y\right) \\ \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \end{cases}$$

Hooke's Law for Plane Strain

$$\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

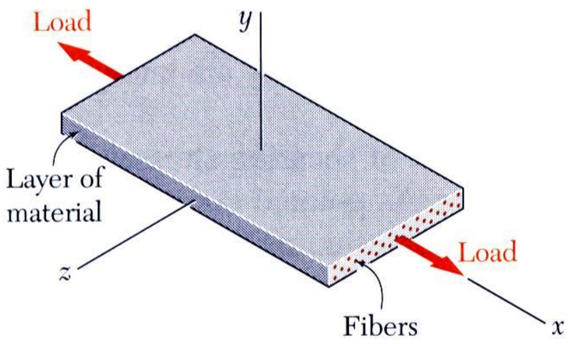
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right]$$

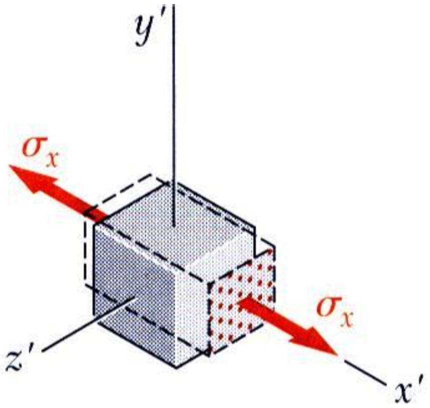
$$0 = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

2.12 **Composite Materials**





- *Fiber-reinforced composite materials* are formed from *lamina(رَقَّقَ المَعْدِنَ)* of fibers(خَيْط) of graphite, glass, or polymers embedded(محشو) in a resin matrix.
- Normal stresses and strains are related by Hooke's Law but with directionally dependent moduli of elasticity,

$$E_x = \frac{\sigma_x}{\varepsilon_x} \quad E_y = \frac{\sigma_y}{\varepsilon_y} \quad E_z = \frac{\sigma_z}{\varepsilon_z}$$

- Transverse contractions are related by directionally dependent values of Poisson's ratio, e.g.,

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} \quad \nu_{xz} = -\frac{\varepsilon_z}{\varepsilon_x}$$

- Materials with directionally dependent mechanical properties are *anisotropic(متباين الخواص)*.

2.12 Composite Materials

$$\varepsilon_x = +\frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y} - \frac{\nu_{zx}\sigma_z}{E_z}$$

$$\varepsilon_y = -\frac{\nu_{xy}\sigma_x}{E_x} + \frac{\sigma_y}{E_y} - \frac{\nu_{zy}\sigma_z}{E_z}$$

$$\varepsilon_z = -\frac{\nu_{xz}\sigma_x}{E_x} - \frac{\nu_{yz}\sigma_y}{E_y} + \frac{\sigma_z}{E_z}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}}$$

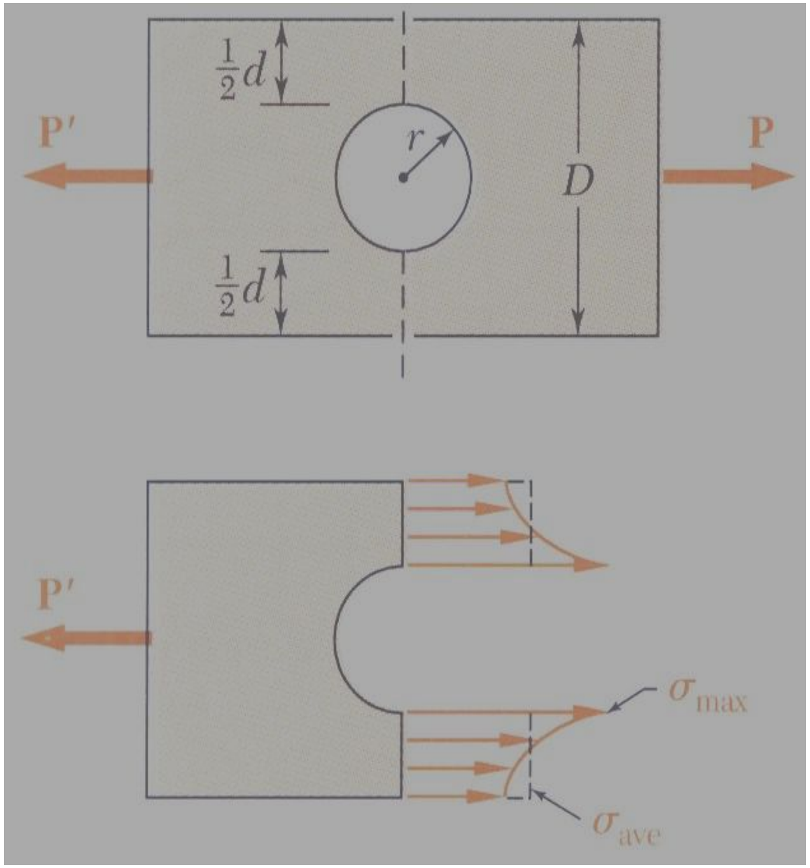
2.12 **Composite Materials**

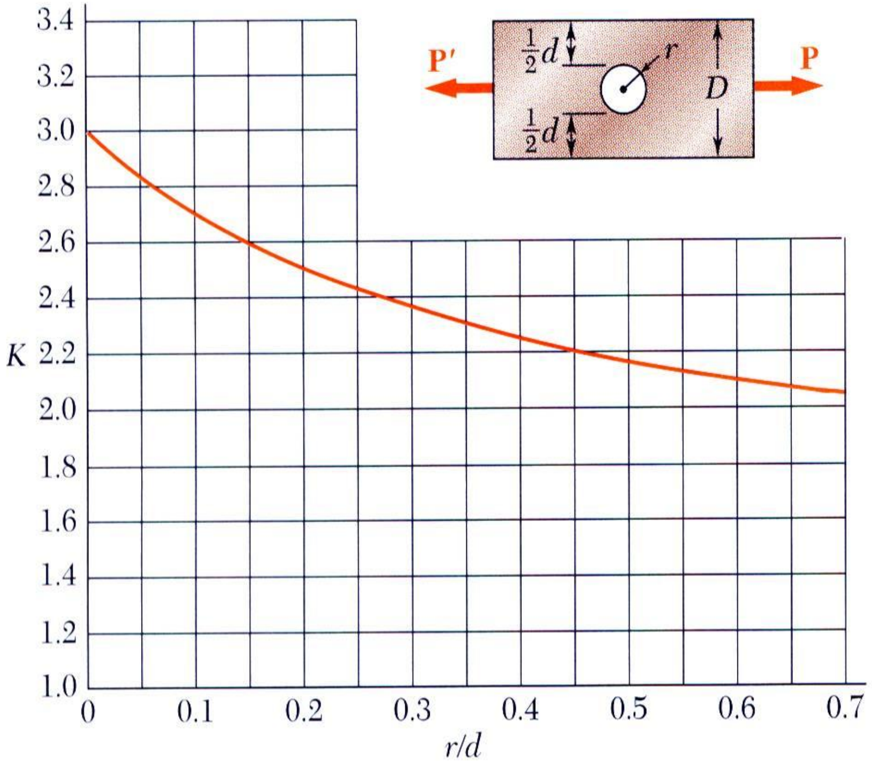
The fact that the three components of strain ϵ_x, ϵ_y, and ϵ_z can be expressed in terms of the normal stresses only and do not depend upon any shearing stresses characterizes *orthotropic materials* and distinguishes them from other anisotropic materials.

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & & & \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & & & \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & & & \\ & & & \frac{1}{G_{23}} & & \\ & & & & \frac{1}{G_{31}} & \\ & & & & & \frac{1}{G_{21}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

2.12 Stress Concentration: Hole

Discontinuities of cross section may result in high localized or *concentrated* stresses.



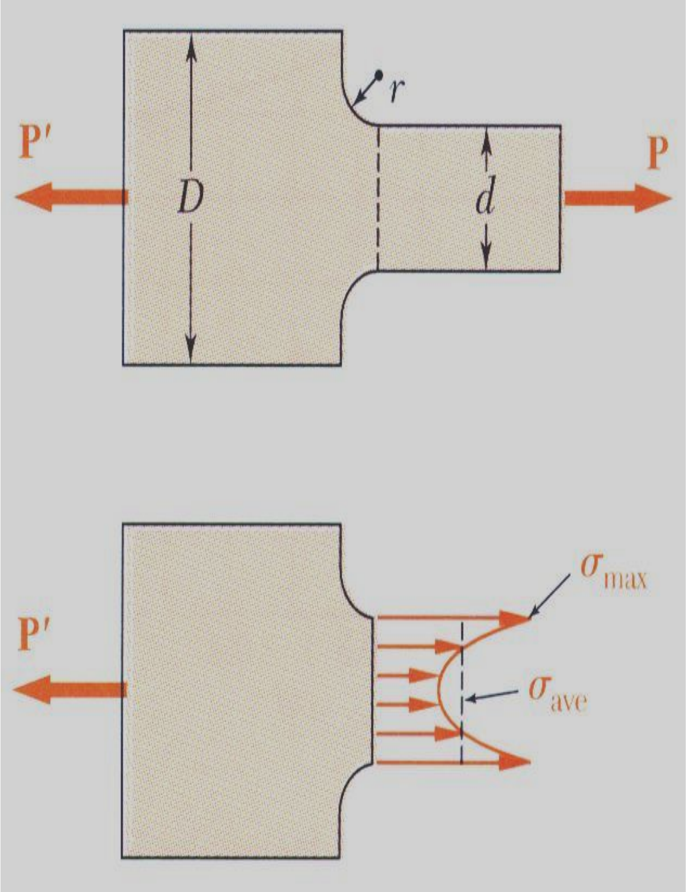


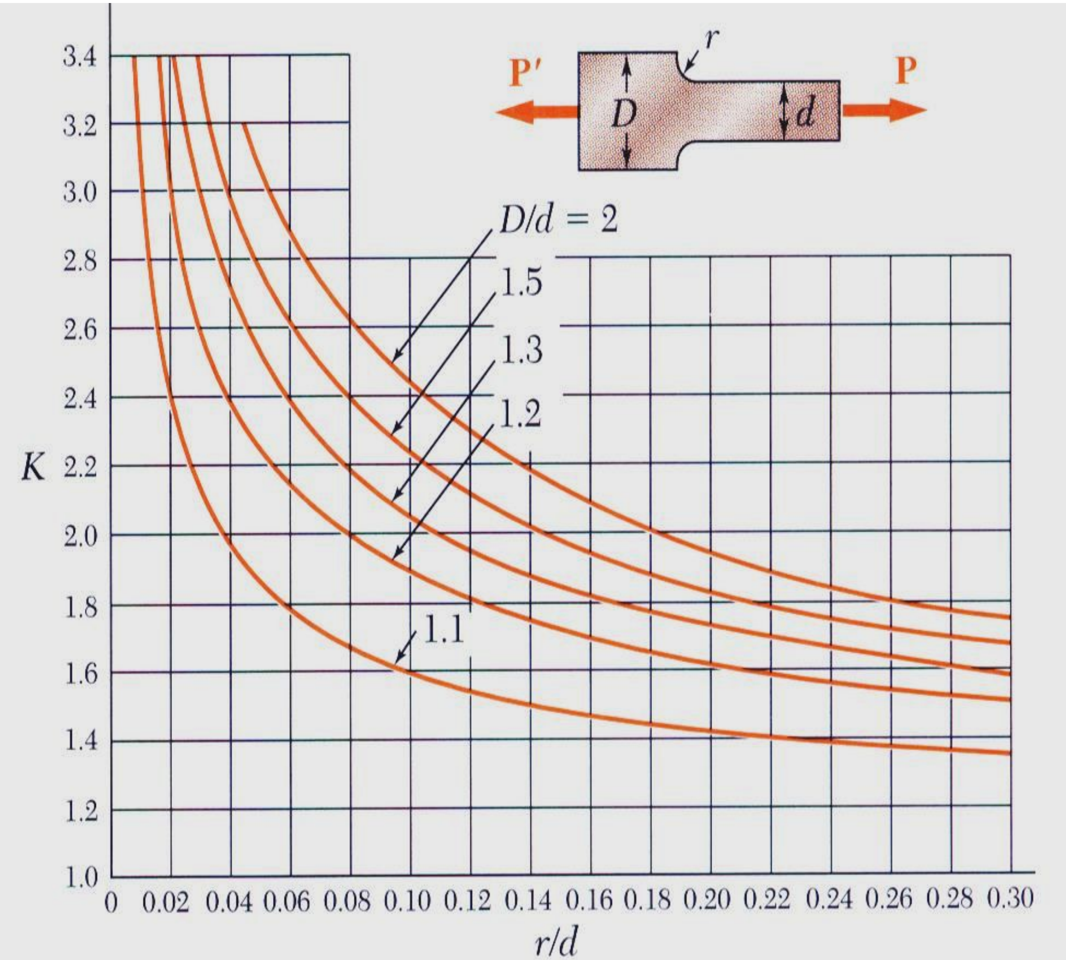
(*a*) Flat bars with holes

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

2.12 Stress Concentration: Hole

Discontinuities of cross section may result in high localized or *concentrated* stresses.

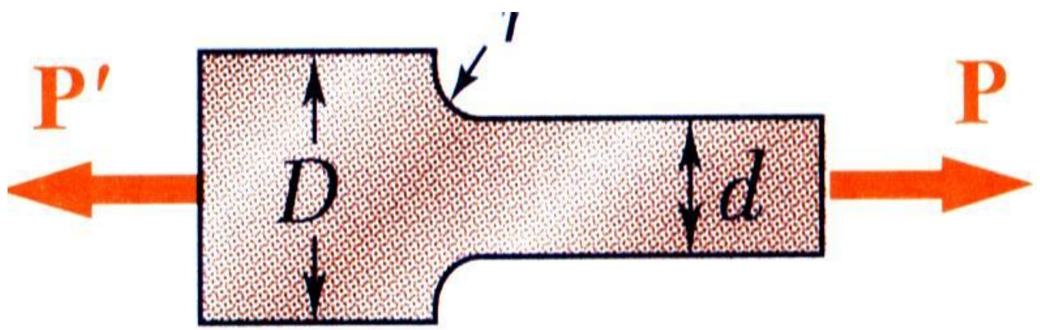




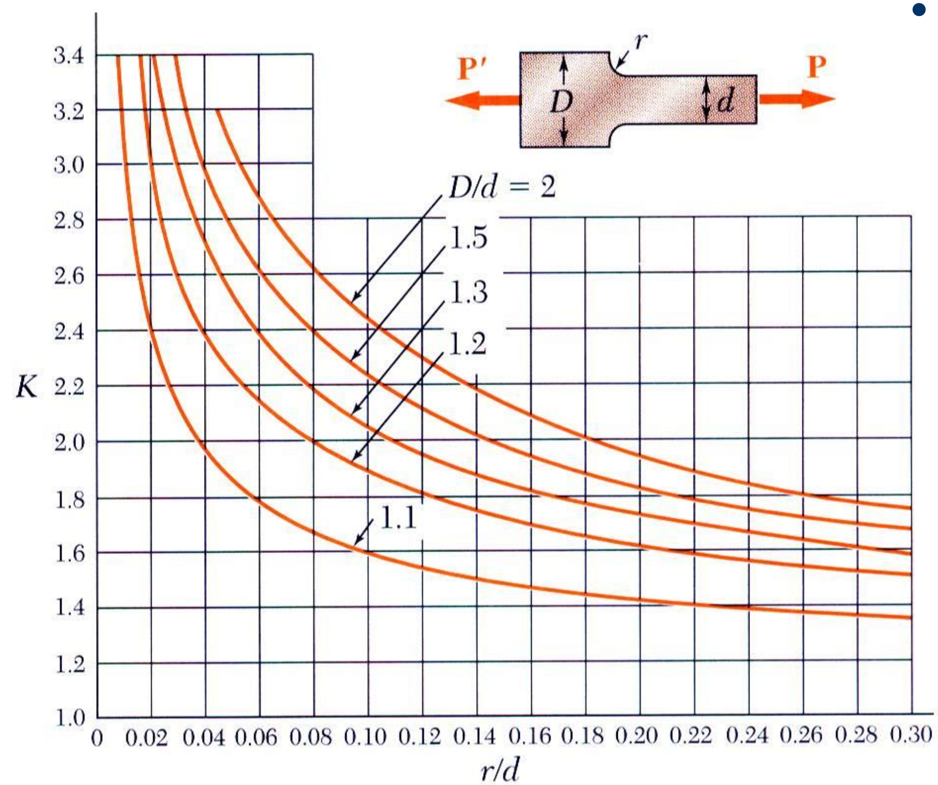
(*b*) Flat bars with fillets

2.12 Stress Concentration: Hole

Example: Determine the largest axial load P that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius r = 8 mm. Assume an allowable normal stress of 165 MPa.



2.12 Stress Concentration: Hole



(*b*) Flat bars with fillets

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64*b*.

$$\frac{D}{d} = \frac{60\,\text{mm}}{40\,\text{mm}} = 1.50 \qquad \frac{r}{d} = \frac{8\,\text{mm}}{40\,\text{mm}} = 0.20$$

$$K = 1.82$$

- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

$$\sigma_{\text{ave}} = \frac{\sigma_{\text{max}}}{K} = \frac{165\,\text{MPa}}{1.82} = 90.7\,\text{MPa}$$

- Apply the definition of normal stre to find the allowable load.

$$P = A\sigma_{ave} = (40\,\text{mm})(10\,\text{mm})(90.7\,\text{MPa})$$

$$= 36.3 \times 10^3\,\text{N}$$

$$\boxed{P = 36.3\,\text{kN}}$$

2.12 Stress Concentration: Hole