

Propositional logic

Irina Prosvirnina

- Propositions
- Compound propositions
- Conditional statements
- Truth tables of compound propositions
- Tautologies and contradictions
- Logical equivalences
- Propositional satisfiability
- Satisfiability problem

Propositions

Our discussion begins with an introduction to the basic building blocks of logic – propositions.

Definition 1

A **proposition** is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Propositions

Example 1

All the following declarative sentences are propositions.

1. Minsk is the capital of Belarus.
 2. Toronto is the capital of Canada.
 3. $1+1=2$.
 4. $2+2=3$.
-

Propositions 1 and 3 are true, whereas 2 and 4 are false.

Propositions

Example 2 Consider the following sentences.

1. What time is it?
 2. Read this carefully.
 3. $x + 1 = 2$.
 4. $x + y = z$.
-

Sentences 1 and 2 are not propositions because they are not declarative sentences.

Sentences 3 and 4 are not propositions because they are neither true nor false.

Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

Propositions

We use letters to denote **propositional variables** (or **statement variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s, \dots$.

The **truth value** of a proposition is true, denoted by T , if it is a true proposition, and the truth value of a proposition is false, denoted by F , if it is a false proposition.

Propositions

The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.

It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

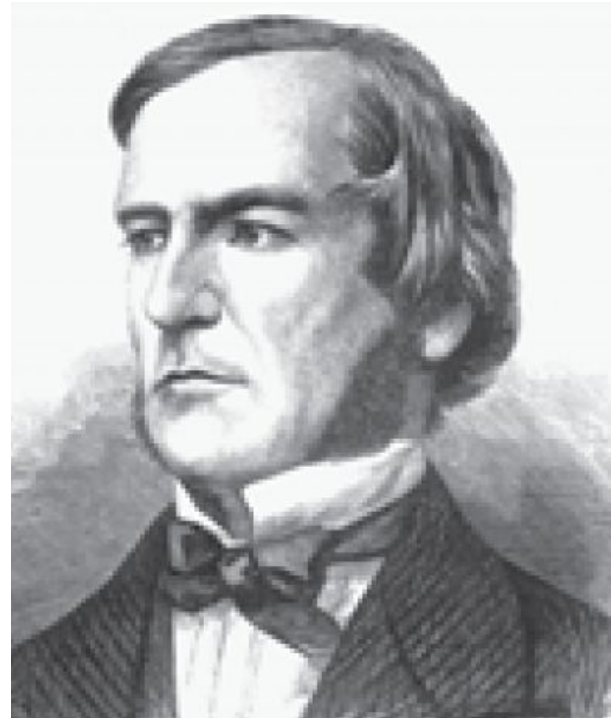


Aristotle
(384 b.c.e.–322 b.c.e.)

Compound propositions

We now turn our attention to methods for producing new propositions from those that we already have.

These methods were discussed by the English mathematician George Boole in 1854 in his book *The Laws of Thought*.



George Boole
(1815–1864)

Compound propositions

Many mathematical statements are constructed by combining one or more propositions.

New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

The negation of a proposition

Definition 2

Let p be a proposition. **The negation** of p , denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement “It is not the case that p .” The proposition $\neg p$ is read “not p .” The truth value of the negation of p , $\neg p$, is the opposite of the truth value of p .

The negation of a proposition

The Truth Table for the
Negation of a Proposition

p	$\neg p$
T	F
F	T

The negation of a proposition

Example 3 Find the negation of the proposition “Vandana’s smartphone has at least 32GB of memory” and express this in simple English.

Solution The negation is “It is not the case that Vandana’s smartphone has at least 32GB of memory.”

This negation can also be expressed as “Vandana’s smartphone does not have at least 32GB of memory” or even more simply as “Vandana’s smartphone has less than 32GB of memory.”

The conjunction of two propositions

Definition 3

Let p and q be propositions. **The conjunction** of p and q , denoted by $p \wedge q$, is the proposition “ p and q ”. The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

The conjunction of two propositions

The Truth Table for the
Conjunction of Two Propositions

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The conjunction of two propositions

Example 4 Find the conjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”

The conjunction of two propositions

Find the conjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”

Solution The conjunction of these propositions, $p \wedge q$, is the proposition “Rebecca’s PC has more than 16 GB free hard disk space, and the processor in Rebecca’s PC runs faster than 1 GHz.”

This conjunction can be expressed more simply as “Rebecca’s PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz.”

For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

The disjunction of two propositions

Definition 4

Let p and q be propositions. The disjunction of p and q , denoted by $p \vee q$, is the proposition “ p or q ”. **The disjunction** $p \vee q$ is false when both p and q are false and is true otherwise.

The disjunction of two propositions

The Truth Table for the Disjunctions of Two Propositions

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The disjunction of two propositions

Example 5 Find the disjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”

The disjunction of two propositions

Find the disjunction of the propositions p and q where p is the proposition “Rebecca’s PC has more than 16 GB free hard disk space” and q is the proposition “The processor in Rebecca’s PC runs faster than 1 GHz.”

Solution The disjunction of p and q , $p \vee q$, is the proposition “Rebecca’s PC has at least 16 GB free hard disk space, or the processor in Rebecca’s PC runs faster than 1 GHz.”

This proposition is true when Rebecca’s PC has at least 16 GB free hard disk space, when the PC’s processor runs faster than 1 GHz, and when both conditions are true. It is false when both of these conditions are false, that is, when Rebecca’s PC has less than 16 GB free hard disk space and the processor in her PC runs at 1 GHz or slower.

The exclusive or

The use of the connective or in a disjunction corresponds to one of the two ways the word or is used in English, namely, as an inclusive or.

A disjunction is true when at least one of the two propositions is true.

The exclusive or

On the other hand, we are using the **exclusive or** when we say “Students who have taken calculus or computer science, but not both, can enroll in this class.”

Here, we mean that students who have taken both calculus and a computer science course cannot take the class. Only those who have taken exactly one of the two courses can take the class.

The exclusive or

Definition 5

Let p and q be propositions. **The exclusive or** of p and q , denoted by $p \square q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The exclusive or

The Truth Table for the Exclusive Or of Two Propositions

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional statements

Definition 6

Let p and q be propositions. **The conditional statement** $p \rightarrow q$ is the proposition “if p , then q ”. The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

In the conditional statement $p \rightarrow q$, p is called **the hypothesis** (or antecedent or premise) and q is called **the conclusion** (or consequence).

Conditional statements

The Truth Table for the
Conditional Statement $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional statements

The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds.

A conditional statement is also called **an implication**.

Conditional statements

Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of the following ways to express this conditional statement:

- “if p , then q ”
- “ p implies q ”
- “ p is sufficient for q ”
- “a sufficient condition for q is p ”
- “ q is necessary for p ”
- “a necessary condition for p is q ”
- “ q follows from p ”
- “ q unless $\neg p$ ”

Conditional statements

Example 6 Let p be the statement “Maria learns discrete mathematics” and q the statement “Maria will find a good job.” Express the statement $p \rightarrow q$ as a statement in English.

Solution From the definition of conditional statements, we see that $p \rightarrow q$ represents the statement “If Maria learns discrete mathematics, then she will find a good job.”

There are many other ways to express this conditional statement in English. Among the most natural of these are: “Maria will find a good job when she learns discrete mathematics.”

“For Maria to get a good job, it is sufficient for her to learn discrete mathematics.”

Converse, contrapositive and inverse

• We can form some new conditional statements starting with a conditional statement $p \rightarrow q$.

In particular, there are three related conditional statements that occur so often that they have special names.

- The proposition $q \rightarrow p$ is called **the converse** of $p \rightarrow q$.
- **The contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
- The proposition $\neg p \rightarrow \neg q$ is called **the inverse** of $p \rightarrow q$.

We will see that of these three conditional statements formed from $p \rightarrow q$, only the contrapositive always has the same truth value as $p \rightarrow q$.

Converse, contrapositive and inverse

Example 7 What are the contrapositive, the converse, and the inverse of the conditional statement “The home team wins whenever it is raining?”

Solution Because “ q whenever p ” is one of the ways to express the conditional statement $p \rightarrow q$, the original statement can be rewritten as “If it is raining, then the home team wins.”

Consequently, **the contrapositive** of this conditional statement is “If the home team does not win, then it is not raining.”

The converse is “If the home team wins, then it is raining.”

The inverse is “If it is not raining, then the home team does not win.”

Biconditionals

We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

Definition 7

Let p and q be propositions. **The biconditional statement** $p \leftrightarrow q$ is the proposition “ p if and only if q .” The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called **bi-implications**.

Biconditionals

The Truth Table for the Biconditional
 $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditionals

Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise.

That is why we use the words “if and only if” to express this logical connective and why it is symbolically written by combining the symbols $\rightarrow$ and $\leftarrow$.

There are some other common ways to express $p \leftrightarrow q$:

- “p is necessary and sufficient for q”
- “if p then q, and conversely”
- “p iff q” (The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation “iff” for “if and only if”.)

Biconditionals

Example 8

Let p be the statement “You can take the flight,” and let q be the statement “You buy a ticket.” Then $p \leftrightarrow q$ is the statement “You can take the flight if and only if you buy a ticket.”

This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

Truth tables of compound propositions

We have now introduced four important logical connectives – conjunctions, disjunctions, conditional statements, and biconditional statements – as well as negations.

We can use these connectives to build up complicated compound propositions involving any number of propositional variables.

Truth tables of compound propositions

We can use truth tables to determine the truth values of these compound propositions.

We use a separate column to find the truth value of each compound expression that occurs in the compound proposition as it is built up.

The truth values of the compound proposition for each combination of truth values of the propositional variables in it is found in the final column of the table.

Truth tables of compound propositions

Example 9 Construct the truth table of the compound proposition $(p \vee \neg q) \wedge (p \vee q)$.

Solution

Because this truth table involves two propositional variables p and q , there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF.

Truth tables of compound propositions

Example 9 Construct the truth table of the compound proposition $(p \vee \neg q) \wedge (p \vee q)$.

Solution

The first two columns are used for the truth values of p and q , respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \vee \neg q$, found in the fourth column.

The fifth column gives the truth value of $p \vee q$.

Finally, the truth value of $(p \vee \neg q) \wedge (p \vee q)$ is found in the last column.

Truth tables of compound propositions

Example 9 Construct the truth table of the compound proposition $(p \vee \neg q) \wedge (p \vee q)$.

The Truth Table of $(p \vee \neg q) \wedge (p \vee q)$					
p	q	$\neg q$	$p \vee \neg q$	$p \vee q$	$(p \vee \neg q) \wedge (p \vee q)$
T	T				
T	F				
F	T				
F	F				

Truth tables of compound propositions

Example 9 Construct the truth table of the compound proposition $(p \vee \neg q) \wedge (p \vee q)$.

The Truth Table of $(p \vee \neg q) \wedge (p \vee q)$					
p	q	$\neg q$	$p \vee \neg q$	$p \vee q$	$(p \vee \neg q) \wedge (p \vee q)$
T	T	F			
T	F	T			
F	T	F			
F	F	T			

Truth tables of compound propositions

Example 9 Construct the truth table of the compound proposition $(p \vee \neg q) \wedge (p \vee q)$.

The Truth Table of $(p \vee \neg q) \wedge (p \vee q)$					
p	q	$\neg q$	$p \vee \neg q$	$p \vee q$	$(p \vee \neg q) \wedge (p \vee q)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	F	T	F
F	F	T	T	F	F

Truth tables of compound propositions

Example 9 Construct the truth table of the compound proposition $(p \vee \neg q) \wedge (p \vee q)$.

The Truth Table of $(p \vee \neg q) \wedge (p \vee q)$					
p	q	$\neg q$	$p \vee \neg q$	$p \vee q$	$(p \vee \neg q) \wedge (p \vee q)$
T	T	F	T	T	
T	F	T	T	F	
F	T	F	F	F	
F	F	T	T	F	

Truth tables of compound propositions

Example 9 Construct the truth table of the compound proposition $(p \vee \neg q) \wedge (p \vee q)$.

The Truth Table of $(p \vee \neg q) \wedge (p \vee q)$					
p	q	$\neg q$	$p \vee \neg q$	$p \vee q$	$(p \vee \neg q) \wedge (p \vee q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of logical operators

• We can construct compound propositions using the negation operator and the logical operators defined so far.

We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied.

For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$.

Precedence of logical operators

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

This table displays the precedence levels of the logical operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.

Precedence of logical operators

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

To reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q , namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q , namely $\neg(p \wedge q)$.

Precedence of logical operators

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$.

Precedence of logical operators

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Finally, it is an accepted rule that the conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than the conjunction and disjunction operators, $\wedge$ and $\vee$.

Consequently, $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$. The conditional operator has precedence over the biconditional operator.

Tautologies and contradictions

Definition 8

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called **a tautology**.

A compound proposition that is always false is called **a contradiction**.

A compound proposition that is neither a tautology nor a contradiction is called **a contingency**.

Tautologies and contradictions

Example 10

We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$.

Examples of a Tautology and a Contradiction			
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Tautologies and contradictions

Example 10

Because $p \vee \neg p$ is always true, it is a tautology.

Because $p \wedge \neg p$ is always false, it is a contradiction.

Examples of a Tautology and a Contradiction			
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical equivalences

Definition 9

The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology.

The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology.

The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

Logical equivalences

One way to determine whether two compound propositions are equivalent is to use a truth table.

In particular, the compound propositions p and q are equivalent if and only if the columns giving their truth values agree.

Logical equivalences

Example 11 Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.						
p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T					
T	F					
F	T					
F	F					

Logical equivalences

Example 11 Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.						
p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T				
T	F	T				
F	T	T				
F	F	F				

Logical equivalences

Example 11 Show that $\neg(p \rightarrow q)$ and $\neg p \wedge q$ are logically equivalent.

Truth Tables for $\neg(p \rightarrow q)$ and $\neg p \wedge q$.						
p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p$	$\neg q$	$\neg p \wedge q$
T	T	T	F			
T	F	T	F			
F	T	T	F			
F	F	F	T			

Logical equivalences

Example 11 Show that $\neg(p \rightarrow q)$ and $\neg p \wedge q$ are logically equivalent.

Truth Tables for $\neg(p \rightarrow q)$ and $\neg p \wedge q$.						
p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p$	$\neg q$	$\neg p \wedge q$
T	T	T	F	F		
T	F	T	F	F		
F	T	T	F	T		
F	F	F	T	T		

Logical equivalences

Example 11 Show that $\neg(p \rightarrow q)$ and $\neg p \wedge q$ are logically equivalent.

Truth Tables for $\neg(p \rightarrow q)$ and $\neg p \wedge q$.						
p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p$	$\neg p \wedge q$	$\neg p \wedge q$
T	T	T	F	F	F	
T	F	T	F	F	T	
F	T	T	F	T	F	
F	F	F	T	T	T	

Logical equivalences

Example 2 Show that $\neg(p \rightarrow q)$ and $\neg p \wedge q$ are logically equivalent.

Truth Tables for $\neg(p \rightarrow q)$ and $\neg p \wedge q$.						
p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p$	$\neg q$	$\neg p \wedge q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Logical equivalences

Example 11 Show that $\neg(p \rightarrow q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \rightarrow q)$ and $\neg p \wedge \neg q$.						
p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Logical equivalences

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.						
p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent.

Logical equivalences

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

This logical equivalence is one of the two De Morgan laws, named after the English mathematician Augustus De Morgan, of the mid-nineteenth century.



Augustus de Morgan
(1806–1871)

Logical equivalences

Example 12 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Solution: We construct the truth table for these compound propositions.

Logical equivalences

Example 12 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T			
T	F			
F	T			
F	F			

Logical equivalences

Example 12 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F		
T	F	F		
F	T	T		
F	F	T		

Logical equivalences

Example 12 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	
T	F	F	F	
F	T	T	T	
F	F	T	T	

Logical equivalences

Example 12 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p$, $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical equivalences

Example 12 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical equivalences

Example 12 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Because the truth values of $\neg p \vee q$ and $p \rightarrow q$ agree, they are logically equivalent.

Logical equivalences

We will now establish a logical equivalence of two compound propositions involving three different propositional variables p , q , and r .

To use a truth table to establish such a logical equivalence, we need eight rows, one for each possible combination of truth values of these three variables.

We symbolically represent these combinations by listing the truth values of p , q , and r , respectively.

These eight combinations of truth values are TTT, TTF, TFT, TFF, FTT, FTF, FFT, and FFF;

we use this order when we display the rows of the truth table.

Logical equivalences

Example 13 Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the distributive law of disjunction over conjunction.

Solution: We construct truth tables for these compound propositions. Because the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree, these compound propositions are logically equivalent.

A Demonstration That $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow (p \rightarrow r)$ Are Logically Equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

A Demonstration That $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow (p \rightarrow r)$ Are Logically Equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$
T	T	T	T				
T	T	F	F				
T	F	T	F				
T	F	F	F				
F	T	T	T				
F	T	F	F				
F	F	T	F				
F	F	F	F				

A Demonstration That $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow (p \rightarrow r)$ Are Logically Equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$
T	T	T	T	T			
T	T	F	F	T			
T	F	T	F	T			
T	F	F	F	T			
F	T	T	T	T			
F	T	F	F	F			
F	F	T	F	F			
F	F	F	F	F			

A Demonstration That $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow (p \rightarrow r)$ Are Logically Equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T		
T	T	F	F	T	T		
T	F	T	F	T	T		
T	F	F	F	T	T		
F	T	T	T	T	T		
F	T	F	F	F	T		
F	F	T	F	F	F		
F	F	F	F	F	F		

A Demonstration That $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow (p \rightarrow r)$ Are Logically Equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	
T	T	F	F	T	T	T	
T	F	T	F	T	T	T	
T	F	F	F	T	T	T	
F	T	T	T	T	T	T	
F	T	F	F	F	T	F	
F	F	T	F	F	F	T	
F	F	F	F	F	F	F	

A Demonstration That $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow (p \rightarrow r)$ Are Logically Equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

A Demonstration That $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow (p \rightarrow r)$ Are Logically Equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

A Demonstration That $p \sqcup (q \sqcup r)$ and $(p \sqcup q) \sqcup (p \sqcup r)$ Are Logically Equivalent.

p	q	r	$q \sqcup r$	$p \sqcup (q \sqcup r)$	$p \sqcup q$	$p \sqcup r$	$(p \sqcup q) \sqcup (p \sqcup r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Because the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree, these compound propositions are logically equivalent.

Logical equivalences

Next table contains some important equivalences.
In these equivalences, T denotes the compound proposition that is always true and F denotes the compound proposition that is always false.

Table 1. Logical Equivalences.

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

Table 1. Logical Equivalences.

Equivalence	Name
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Logical equivalences

• **Boolean algebra** of propositions is a set P of all propositions with two binary operations: conjunction ($\vee$) and disjunction ($\wedge$), logical constants T and F , and negation operator ($\neg$) that satisfies the identity, complement, associative, commutative, and distributive laws.

Logical equivalences

We also display some useful equivalences for compound propositions involving conditional statements and biconditional statements in Tables 2 and 3, respectively.

Table 2. Logical Equivalences Involving Conditional Statements.

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Table 3. Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Using De Morgan's Laws

Example 13 Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution: Let p be "Miguel has a cellphone" and q be "Miguel has a laptop computer." Then "Miguel has a cellphone and he has a laptop computer" can be represented by $p \wedge q$. By the first of De Morgan's laws, $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$.

Consequently, we can express the negation of our original statement as "Miguel does not have a cellphone **or** he does not have a laptop computer."

Using De Morgan's Laws

Example 13 Use De Morgan's laws to express the negations of "Miguel has a cellphone and he has a laptop computer" and "Heather will go to the concert or Steve will go to the concert."

Solution: Let r be "Heather will go to the concert" and s be "Steve will go to the concert." Then "Heather will go to the concert or Steve will go to the concert" can be represented by $r \vee s$. By the second of De Morgan's laws, $\neg(r \vee s)$ is equivalent to $\neg r \wedge \neg s$.

Consequently, we can express the negation of our original statement as "Heather will not go to the concert **and** Steve will not go to the concert."

Constructing new logical equivalences

The logical equivalences in Table 1, as well as any others that have been established (such as those shown in Tables 2 and 3), can be used to construct additional logical equivalences.

The reason for this is that a proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.

Constructing new logical equivalences

This technique is illustrated in Examples 14 – 16, where we also use the fact that if p and q are logically equivalent and q and r are logically equivalent, then p and r are logically equivalent.

Constructing new logical equivalences

Example 14 Show that $\neg(p \vee q)$ and $p \wedge \neg q$ are logically equivalent.

Solution: We will establish this equivalence by developing a series of logical equivalences, using one of the equivalences in Table 1 at a time, starting with $\neg(p \vee q)$ and ending with $p \wedge \neg q$.

Constructing new logical equivalences

Example 14 Show that $\neg(p \vee q)$ and $p \wedge \neg q$ are logically equivalent.

Solution: We have the following equivalences.

$$\neg(p \vee q) \equiv \neg(\neg p \wedge q) \text{ -- by Example 12}$$

$$\equiv \neg(\neg p) \wedge \neg q \text{ -- by the second De Morgan law}$$

$$\equiv p \wedge \neg q \text{ -- by the double negation law}$$

Constructing new logical equivalences

Example 15 Show that $\neg(p \rightarrow (\neg p \rightarrow q))$ and $(\neg p \rightarrow \neg q)$ are logically equivalent by developing a series of logical equivalences.

Solution:

We will use one of the equivalences in Table 1 at a time, starting with $\neg(p \rightarrow (\neg p \rightarrow q))$ and ending with $(\neg p \rightarrow \neg q)$.

(Note: we could also easily establish this equivalence using a truth table.)

Constructing new logical equivalences

Example 15 Show that $\neg(p \rightarrow (\neg p \rightarrow q))$ and $(\neg p \rightarrow \neg q)$ are logically equivalent by developing a series of logical equivalences.

Solution: We have the following equivalences.

$$\begin{aligned}\neg(p \rightarrow (\neg p \rightarrow q)) &\equiv \neg p \rightarrow \neg(\neg p \rightarrow q) \\ &\equiv \neg p \rightarrow (\neg(\neg p) \wedge \neg q) \\ &\equiv \neg p \rightarrow (p \wedge \neg q) \\ &\equiv (\neg p \wedge p) \wedge (\neg p \wedge \neg q) \\ &\equiv F \wedge (\neg p \wedge \neg q) \\ &\equiv (\neg p \wedge \neg q) \wedge F \\ &\equiv (\neg p \wedge \neg q)\end{aligned}$$

Constructing new logical equivalences

Example 16 Show that $(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow q)$ is a tautology.

Solution:

$$\begin{aligned}(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow q) &\leftrightarrow \neg (p \leftrightarrow q) \vee (p \leftrightarrow q) \\&\leftrightarrow (\neg p \vee \neg q) \vee (p \vee q) \\&\leftrightarrow (\neg p \vee p) \vee (\neg q \vee q) \\&\leftrightarrow T \vee T \\&\leftrightarrow T\end{aligned}$$

Propositional satisfiability

Definition 10 A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.

When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**.

Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

Propositional satisfiability

Definition 11

When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable;
such an assignment is called **a solution** of this particular satisfiability problem.

Propositional satisfiability

However, to show that a compound proposition is unsatisfiable, we need to show that every assignment of truth values to its variables makes it false.

Although we can always use a truth table to determine whether a compound proposition is satisfiable, it is often more efficient not to, as Example 17 demonstrates.

Propositional satisfiability

Example 17 Determine whether each of the compound propositions

$$(p \vee \neg q) \vee (q \vee \neg r) \vee (r \vee \neg p),$$

$$(p \vee q \vee r) \vee (\neg p \vee \neg q \vee \neg r),$$

$$(p \vee \neg q) \vee (q \vee \neg r) \vee (r \vee \neg p) \vee (p \vee q \vee r) \vee (\neg p \vee \neg q \vee \neg r)$$

is satisfiable.

Propositional satisfiability

Example 17

Solution:

- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is satisfiable
 $(p = T, q = T, r = T);$
- $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable
 $(p = T, q = F, r = T);$
- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$
is unsatisfiable (why?).

Satisfiability problem

Many problems, in diverse areas such as

- robotics,
- software testing,
- computer-aided design,
- machine vision,
- integrated circuit design,
- computer networking,
- genetics,

can be modeled in terms of propositional satisfiability.

In particular, we will show how to use propositional satisfiability to model Sudoku puzzles.

Sudoku 9□9

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

A Sudoku puzzle is represented by a 9×9 grid made up of nine 3×3 subgrids, known as blocks.

For each puzzle, some of the 81 cells, called givens, are assigned one of the numbers 1,2,...,9, and the other cells are blank.

Sudoku 9□9

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

The puzzle is solved by assigning a number to each blank cell so that every row, every column, and every one of the nine 3×3 blocks contains each of the nine possible numbers.

Sudoku 9□9

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Exercise Construct a compound proposition that asserts that every cell of a 9×9 Sudoku puzzle contains at least one number.